Topics for Midterm III

You should know the material in chapter 5 (5.1–5.7) of the text. In particular, we emphasized the following topics.

1. A knowledge of matrices, matrix multiplication and addition, column vectors, determinants of square matrices, the identity matrix.
2. Linear systems of ODEs, and the ability to encode a linear system using the language of matrices and column vectors.
3. Linear systems arising from other areas of mathematics or physics/engineering; "real-world" examples.
4. Non-homogeneous and homogeneous linear systems.
5. Superposition principle: General solution of a non-homogeneous linear system is the sum of the general solution to the associated homogeneous system and a particular solution of the non-homogeneous system.
6. Solving simple linear systems by elimination (operator determinant method).
7. Solving homogeneous linear systems by the eigenvalue–eigenvector method.
   - Compute eigenvectors of a square matrix via the characteristic equation.
   - Find eigenvectors $\mathbf{v}_i$ corresponding to eigenvalues λ_i.
   - $\mathbf{x}_i = e^{\lambda_i t}\mathbf{v}_i$ are independent solutions.
8. Working with complex eigenvalues and eigenvectors.
9. Working with generalized eigenvectors.
10. A phase plane portrait gallery for systems with 2×2 matrices.
11. Mechanical vibration examples, and systems of the form $\frac{d^2\mathbf{X}}{dt^2} = A\mathbf{X}$.
12. The matrix $\Phi(t)$ of fundamental solutions to a homogeneous linear system and its relationship with the matrix exponential e^{At}.
13. Alternative methods of computing matrix exponentials for special matrices.
14. Using the variation of parameters method to solve non-homogenous linear systems.

Practice Problems

1. For each of the three matrices A below, first find e^{At}, and then solve the initial value problem

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} \qquad \mathbf{x}(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \qquad A = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \qquad A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

2. Solve the homogeneous linear system

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \mathbf{x}$$

and use this solution to write down a fundamental matrix $\Phi(t)$.
Use $\Phi(t)$ to compute e^{At} where $A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$.

3. Find the general solution to the following system.

$$x'' = -10x + 2y \qquad y'' = 3x - 15y$$

4. Find the general solution to the linear system

$$\mathbf{x}' = \begin{pmatrix} 5 & -1 & 1 \\ 1 & 3 & 0 \\ -3 & 2 & 1 \end{pmatrix} \mathbf{x}$$

where the eigenvalues are $\lambda = 3, 3, 3$.

5. Find the general solution to the linear system

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} -3 & 5 & -5 \\ 3 & -1 & 3 \\ 8 & -8 & 10 \end{pmatrix} \mathbf{x}$$

where the eigenvalues are 2, 2, 2.

6. Use variation of parameters to find a particular solution to the IVP

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} t\cos 2t \\ t\sin 2t \end{pmatrix}$$

where $\mathbf{x}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

7. Find the general solution of the system

$$\mathbf{x}' = \begin{pmatrix} 4 & -3 \\ 3 & 4 \end{pmatrix} \mathbf{x}$$

8. Know the various phase plane pictures; corresponding to various eigenvalues/eigenvectors of the 2×2 matrix.