Q1]...[25 points] Consider the linear system

$$\frac{dx}{dt} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \mathrm{x}$$

Find the eigenvalues of $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ and corresponding eigenvectors.

$$\det \begin{pmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix} = 0 \Rightarrow (1-\lambda)^2 - (2)^2 = 0$$

$$\lambda^2 - 2\lambda - 3 = 0$$
$$(\lambda-3)(\lambda+1) = 0$$
$$\lambda = 3 \quad \lambda = -1$$

$\boxed{\lambda = 3}$

$$\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad -2x+2y=0$$
$$y = x$$

$\Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is a 3-eigenvector

$\boxed{\lambda = -1}$

$$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad 2x+2y=0$$
$$y = -x$$

$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is a (-1)-eigenvector.

Find the general solution of the linear system above.

$$\boxed{\vec{X}(t) = c_1 e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

for real numbers (parameters) c_1 & c_2.

Gen. Sol$^{=}$

Mid III Solutions

Q2]... **[25 points]** Consider the linear system

$$\frac{d\mathrm{x}}{dt} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \mathrm{x}$$

Find the eigenvalues of $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ and corresponding eigenvectors.

$$\det\begin{pmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{pmatrix} = 0 \implies (1-\lambda)^2 - (-1)(1) = 0 \implies \lambda^2 - 2\lambda + 2 = 0$$

$$\lambda = \frac{2 \pm \sqrt{(2)^2 - 4(2)}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$$

$\boxed{\lambda = 1+i}$

$$\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \begin{matrix} -ix - y = 0 \\ y = -ix \end{matrix}$$

$\begin{pmatrix} 1 \\ -i \end{pmatrix}$ is a $(1+i)$-eigenvector

$\boxed{\lambda = 1-i}$

$$\begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \begin{matrix} ix - y = 0 \\ y = ix \end{matrix}$$

$\begin{pmatrix} 1 \\ i \end{pmatrix}$ is a $(1-i)$-eigenvector.

Note that the e-vectors are conjugates too

Find the general solution of the linear system above.

Euler's identity

$$e^{(1+i)t}\begin{pmatrix} 1 \\ -i \end{pmatrix} = e^{t} e^{it}\begin{pmatrix} 1 \\ -i \end{pmatrix} = e^{t}(\cos t + i \sin t)\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$= \begin{pmatrix} e^t \cos t \\ e^t \sin t \end{pmatrix} + i \begin{pmatrix} e^t \sin t \\ -e^t \cos t \end{pmatrix}$$

These form independent sol$^{\underline{n}}$s. Thus the general sol$^{\underline{n}}$ is

$$\boxed{\vec{X}(t) = c_1 e^t \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} + c_2 e^t \begin{pmatrix} \sin t \\ -\cos t \end{pmatrix}}$$

for c_1, c_2 Real numbers (parameters)

Q3]... **[25 points]** Compute the exponential e^{At} of the following two matrices.

$$A = \begin{pmatrix} 2 & 2 \\ -2 & -2 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 2 & 2 \\ -2 & -2 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ -2 & -2 \end{pmatrix} = \begin{pmatrix} 4-4 & 4-4 \\ -4+4 & -4+4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$$

$$e^{At} = I + At + A^2\frac{t^2}{2} + \cdots = I + At + 0 + 0 + \cdots$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 2t & 2t \\ -2t & -2t \end{pmatrix}$$

$$= \underline{\underline{\begin{pmatrix} 1+2t & 2t \\ -2t & 1-2t \end{pmatrix}}}$$

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

because $I\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} I$

$$e^{At} = e^{\left(I + \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}\right)t} = e^{It + \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}t} = e^{It}\, e^{\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}t}$$

$$= \begin{pmatrix} e^t & 0 \\ 0 & e^t \end{pmatrix}\left[I + \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}t + \cancel{\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}^2}^{\,0}\frac{t^2}{2} + \cancel{\cdots}^{\,0} \right]$$

$$= \begin{pmatrix} e^t & 0 \\ 0 & e^t \end{pmatrix}\begin{pmatrix} 1 & 2t \\ 0 & 1 \end{pmatrix} = \underline{\underline{\begin{pmatrix} e^t & 2te^t \\ 0 & e^t \end{pmatrix}}}$$

Suppose that you are told that $e^{Bt} = \begin{pmatrix} e^{3t} & 0 \\ 5te^{3t} & e^{3t} \end{pmatrix}$. Find the solution to the initial value problem

$$\frac{d\mathrm{x}}{dt} = B\mathrm{x}, \qquad \mathrm{x}(0) = \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

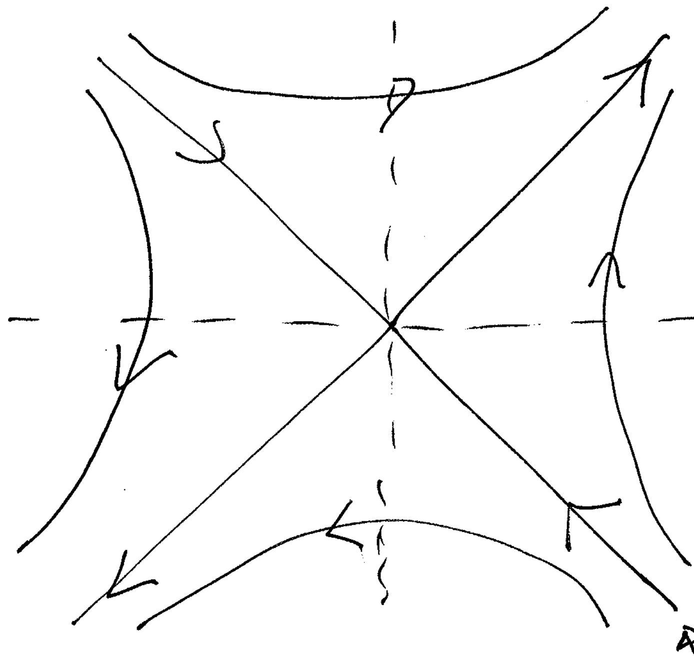
Solution is $\vec{X}(t) = e^{Bt}\vec{X}(0)$

$$= \begin{pmatrix} e^{3t} & 0 \\ 5te^{3t} & e^{3t} \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 3e^{3t} \\ 15te^{3t} + 4e^{3t} \end{pmatrix}}}$$

Q4]... [**25 points**] Draw a sketch of the solution curves (phase plane portrait) of the linear system $\mathbf{x}' = A\mathbf{x}$ where A is a 2×2 matrix with the following eigenvalues/eigenvectors. Give some reasons to justify your diagrams.

1. A has one eigenvalue equal to 2 with eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and a second eigenvalue equal to -2 and eigenvector $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

$e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow$ expanding as $t \to \infty$

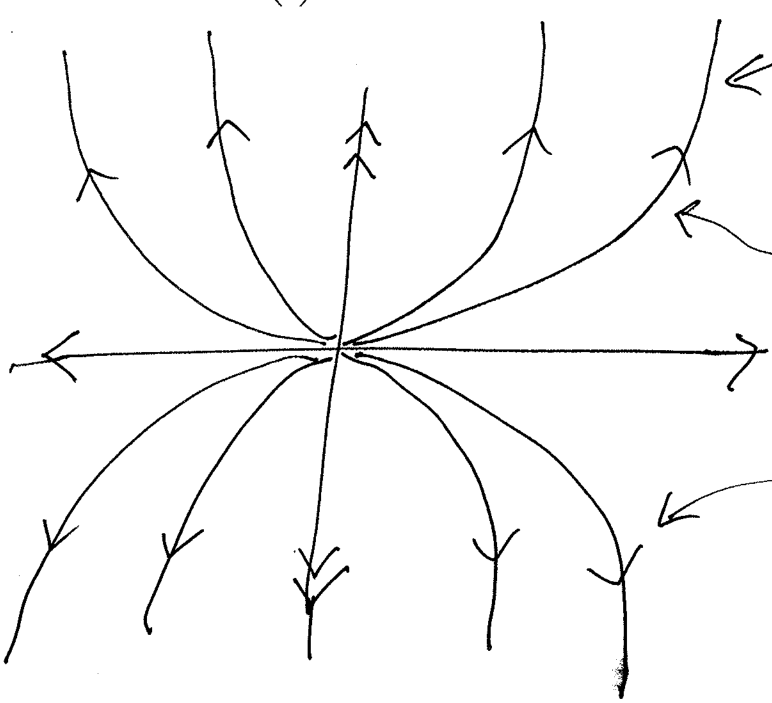


$x = e^{2t}$, $y = e^{-2t}$ $\Rightarrow$ $xy = 1$ ← hyperbolas

but with axes rotated by $\pi/4$

$e^{-2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow$ contracting as $t \to \infty$.

2. A has one eigenvalue equal to 2 with eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and a second eigenvalue equal to 5 and eigenvector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.



$y = e^{5t}$

$x = e^{2t}$

$y = (e^{2t})^{5/2} = x^{5/2}$

$y = x^{5/2}$ shaped curves