Sp'16: MATH 3113–002 **Introduction to ODEs** **Noel Brady**

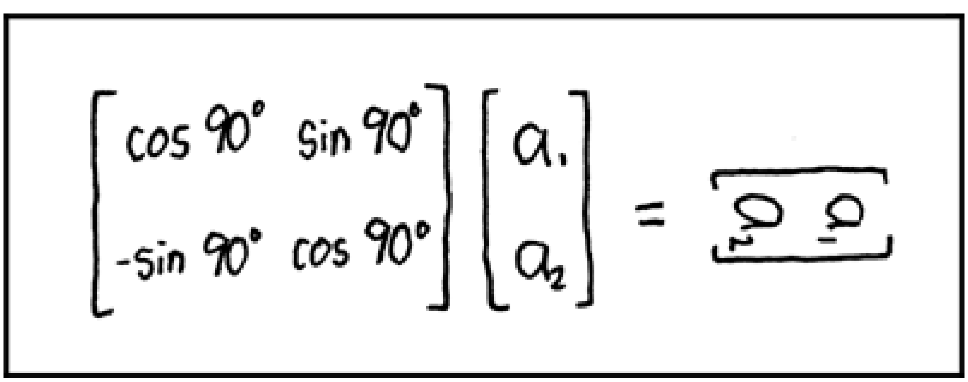
Friday 04/22/2016 Midterm III 50 minutes

Name: ____________________ Student ID: ____________________

Instructions.

1. Attempt all questions.
2. Do not write on back of exam sheets. Extra paper is available if you need it.
3. **Show all the steps of your work clearly. A final answer on its own is not enough for full points.**

Question	Points	Your Score
Q1	25	
Q2	25	
Q3	25	
Q4	25	
TOTAL	100	



Miscellaneous expressions and definitions.

1. **Trig Addition, Half Angle.**

   $\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$ $\quad \cos(2A) = \cos^2(A) - \sin^2(A)$

   $\cos(2A) = 2\cos^2(A) - 1$ $\quad \cos(2A) = 1 - 2\sin^2(A)$ $\quad \sin^2(x) = (1 - \cos(2x))/2$

   $\cos^2(x) = (1 + \cos(2x))/2$ $\quad \sin(2x) = 2\sin(x)\cos(x)$

   $\sin(A \pm B) = \sin(A)\cos(B) \pm \cos(A)\sin(B)$

2. **Hyperbolic.**

   $\sinh(x) = \frac{1}{2}(e^x - e^{-x})$ $\quad \cosh(x) = \frac{1}{2}(e^x + e^{-x})$

3. **Integration by Parts.**

   $\int u\,dv = uv - \int v\,du$

4. **Integration by substitution.**

$$\int f(u(x))\frac{du}{dx}\,dx = \int f(u)\,du$$

5. **Inverse Trig.**

   $\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$ $\quad \frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$ $\quad \int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}(\frac{x}{a})$

6. **Trig Substitutions.**

   For $\sqrt{a^2 - x^2}$ use $x = a\sin(\theta)$
   For $\sqrt{a^2 + x^2}$ use $x = a\tan(\theta)$
   For $\sqrt{x^2 - a^2}$ use $x = a\sec(\theta)$

7. **Some integrals.**

$$\int \frac{dx}{x} = \ln|x| + C \qquad \int \tan(x)\,dx = \ln|\sec(x)| + C$$

$$\int \sec(x)\,dx = \ln|\sec(x) + \tan(x)| + C$$

8. **Homogeneous systems.** An nth order homogeneous system is an equation of the form

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} \qquad (+)$$

   where $\mathbf{x} = \mathbf{x}(t)$ is an $n \times 1$ column vector of functions of t, and A is an $n \times n$ matrix whose entries are functions of t. If the entries of A are all constants, then the system is called constant coefficient.

   A general solution of (+) is of the form

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + \cdots + c_n\mathbf{x}_n(t)$$

   where the c_i are constants (parameters) and the $\mathbf{x}_i(t)$ are n linearly independent column vectors.

9. **Linear systems.** A general linear system is an equation of the form

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} + \mathbf{f}(t) \qquad (*)$$

where $\mathbf{x} = \mathbf{x}(t)$ is an $n \times 1$ column vector of functions of t, A is an $n \times n$ matrix whose entries are functions of t, and $\mathbf{f}(t)$ is a column vector of functions of t. If the entries of A are all constants, then the system is called constant coefficient.

The general solution of $(*)$ is of the form

$$\mathbf{x}(t) = \mathbf{x}_p(t) + \mathbf{x}_h(t)$$

where $\mathbf{x}_p(t)$ is a particular solution of $(*)$ and $\mathbf{x}_h(t)$ is the general solution of the associated homogeneous equation $(+)$.

10. **Eigenvalue-eigenvector method.** The solutions of $(+)$ in the constant-coefficient case are obtained by (1) solving $\det(A - \lambda I) = 0$ to find the eigenvalues λ of A; (2) for each eigenvalue λ solving the equation $(A - \lambda I)\mathbf{v} = \mathbf{0}$ for eignevectors $\mathbf{v}$; and (3) constructing solutions of the form $e^{\lambda t}\mathbf{v}$.

One has to take care with (1) complex eigenvalues/eigenvectors, and with (2) repeated eigenvalues that may give rise to generalized eigenvectors and solutions of the form $e^{\lambda t}(t\mathbf{v}_1 + \mathbf{v}_2)$ etc.

The case of complex eigenvectors will be greatly simplified if you use Euler's identity

$$e^{i\theta} = \cos\theta + i\sin\theta$$

11. **Fundamental matrix and exponential matrix.** Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ be n linearly independent solutions to $(+)$. Then writing these as column vectors of an $n \times n$ matrix gives the fundamental matrix $\Phi(t)$. Note that

$$\Phi(t)(\Phi(0))^{-1} = e^{At}$$

where the exponential matrix e^{At} is defined by the power series

$$e^{At} = I + At + \frac{A^2t^2}{2} + \cdots$$

For certain matrices A it is easier to directly compute e^{At} directly from the power series definition.

12. **Undetermined coefficients.** Let $\Phi(t)$ be a fundamental matrix for the homogeneous system $(+)$. Then a particular solution of $(*)$ is obtained by the formula

$$\mathbf{x}_p(t) = \Phi(t) \int (\Phi(t))^{-1}\mathbf{f}(t)\, dt$$

Q1]. . . [25 points] Consider the linear system

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \mathbf{x}$$

Find the eigenvalues of $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ and corresponding eigenvectors.

Find the general solution of the linear system above.

Q2]. . . [25 points] Consider the linear system

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \mathbf{x}$$

Find the eigenvalues of $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ and corresponding eigenvectors.

Find the general solution of the linear system above.

Q3]...[25 points] Compute the exponential e^{At} of the following two matrices.

$$A = \begin{pmatrix} 2 & 2 \\ -2 & -2 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

Suppose that you are told that $e^{Bt} = \begin{pmatrix} e^{3t} & 0 \\ 5te^{3t} & e^{3t} \end{pmatrix}$. Find the solution to the initial value problem

$$\frac{d\mathbf{x}}{dt} = B\mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

Q4]. . . [25 points] Draw a sketch of the solution curves (phase plane portrait) of the linear system $\mathbf{x}' = A\mathbf{x}$ where A is a 2×2 matrix with the following eigenvalues/eigenvectors. Give some reasons to justify your diagrams.

1. A has one eigenvalue equal to 2 with eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and a second eigenvalue equal to -2 and eigenvector $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

2. A has one eigenvalue equal to 2 with eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and a second eigenvalue equal to 5 and eigenvector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.