Q1]. . . [25 points] Consider the following ODE.

$$(D^2 - 4D + 5)(D^2 + 4)^2(D - 3)^3 D^4 y = 0$$

What is its order?

order = highest exponent of D $= 2 + 2(2) + 3 + 4 = \boxed{13}$

How many independent solutions should it have?

SAME AS ORDER ! i.e. $\boxed{13}$

Find the general solution to the ODE above.

Characteristic equation is

$$(r^2 - 4r + 5)(r^2+4)^2 (r-3)^3 r^4 = 0$$

$r = \dfrac{-(-4) \pm \sqrt{(-4)^2 - 4(5)}}{2} = \dfrac{4 \pm 2i}{2} = 2 \pm i$

$e^{2x}\cos x, \; e^{2x}\sin x$

$r^2 = -4$

$r = \pm 2i$

(multiplicity = 2)

$\cos 2x, \; \sin 2x,$

$x\cos 2x, \; x \sin 2x$

$r = 3$

(Multiplicity = 3)

$e^{3x}, \; xe^{3x}, \; x^2 e^{3x}$

$r = 0$

(Multiplicity = 4)

$e^{0x} = 1$

$\Downarrow$

$1, x, x^2, x^3$

$\Downarrow$

General solution (will have 13 terms) is

$$\boxed{\begin{aligned} y = {} & C_1 e^{2x}\cos x + C_2 e^{2x} \sin x + C_3 \cos 2x + C_4 \sin 2x + C_5 x \cos 2x + C_6 x \sin 2x \\ & + C_7 e^{3x} + C_8 x e^{3x} + C_9 x^2 e^{3x} + C_{10} + C_{11} x + C_{12} x^2 + C_{13} x^3 . \end{aligned}}$$

Q1 comments

(1) Order of ODE $\equiv$ # independent solutions

$\parallel\!\!|$

THESE NUMBERS ARE ALWAYS THE SAME!

$\to$ number of terms in general solution.

all 3 numbers were 13 for this particular Qn,

(2) A factored linear differential operator

$$(D^2 - 4D + 5)(D^2 + 4)^2 \text{ etc.}$$

implies that the characteristic equation comes "pre-factored" ...

$$(r^2 - 4r + 5)(r^2 + 4)^2 \text{ etc} = 0$$

& so roots are (almost) immediate to find ...

$r^2 - 4r + 5 = 0$

$\Downarrow$

$r = \dfrac{4 \pm \sqrt{16 - 20}}{2}$

$= 2 \pm i$

$r^2 = -4$

$r = \pm 2i$

$\uparrow$

Mult. $= 2$

etc. . . .

Q2]... **[25 points]** Show that the following four functions are linearly independent on $(-\infty, \infty)$.

$$\sin(x), \qquad \cos(x), \qquad e^{3x}, \qquad xe^{3x}$$

<u>Linear independence</u> means that the <u>only</u> solution to

$$c_1 \sin x + c_2 \cos x + c_3 e^{3x} + c_4 x e^{3x} = 0 \quad \leftarrow \text{constant function } 0$$

is the trivial one, $c_1 = c_2 = c_3 = c_4 = 0$.

$$c_1 \sin x + c_2 \cos x + c_3 e^{3x} + c_4 x e^{3x} = 0$$

$$\downarrow \tfrac{d}{dx}$$

$$c_1 \cos x - c_2 \sin x + 3c_3 e^{3x} + c_4 e^{3x} + 3c_4 x e^{3x} = 0$$

$$\downarrow \tfrac{d}{dx}$$

$$-c_1 \sin x - c_2 \cos x + 9 c_3 e^{3x} + 6 c_4 e^{3x} + 9 c_4 x e^{3x} = 0$$

$$\downarrow \tfrac{d}{dx}$$

$$-c_1 \cos x + c_2 \sin x + 27 c_3 e^{3x} + 27 c_4 e^{3x} + 27 c_4 x e^{3x} = 0$$

Now let $x = 0$ to get

$c_2 = -c_3$

$$c_2 + c_3 = 0$$
$$c_1 + 3c_3 + c_4 = 0$$
$$-c_2 + 9c_3 + 6c_4 = 0$$
$$-c_1 + 27 c_3 + 27 c_4 = 0$$

$\Rightarrow$

$$c_1 + 3c_3 + c_4 = 0$$
$$10 c_3 + 6 c_4 = 0$$
$$-c_1 + 27 c_3 + 27 c_4 = 0$$

Add

$\times 3$

$$30 c_3 + 28 c_4 = 0$$
$$30 c_3 + 18 c_4 = 0$$

(subtract)

$$10 c_4 = 0$$
$$\Rightarrow \boxed{c_4 = 0}$$

$$30 c_3 + 28(0) = 0$$
$$\Rightarrow 30 c_3 = 0$$
$$\Rightarrow \boxed{c_3 = 0}$$

$$c_1 + 3(0) + (0) = 0$$
$$\Rightarrow \boxed{c_1 = 0}$$

$$\& \quad c_2 = -c_3 = -0 = 0 \quad \Rightarrow \boxed{c_2 = 0}$$

Therefore $\sin x$, $\cos x$, e^{3x}, xe^{3x} are linearly independent.

Comment on Q2.

Some of you computed the Wronskian $\begin{vmatrix} f & g & h & k \\ f' & g' & h' & k' \\ f'' & g'' & h'' & k'' \\ f''' & g''' & h''' & k''' \end{vmatrix}$

This is a valid approach .... but please, please move to <u>numbers</u> quickly!

DO **NOT** COMPUTE THIS IN PUBLIC!

$$W(x) = \begin{vmatrix} \sin x & \cos x & e^{3x} & xe^{3x} \\ \cos x & -\sin x & 3e^{3x} & e^{3x} + 3xe^{3x} \\ -\sin x & -\cos x & 9e^{3x} & 6e^{3x} + 9xe^{3x} \\ -\cos x & \sin x & 27e^{3x} & 27e^{3x} + 27xe^{3x} \end{vmatrix}$$

We'll show $W(x) \neq 0$ by showing $W(0) \neq 0$.

Let $x = 0$ ....

$$W(0) = \begin{vmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 3 & 1 \\ 0 & -1 & 9 & 6 \\ -1 & 0 & 27 & 27 \end{vmatrix}$$

$$= \underbrace{0 \begin{vmatrix} 0 & 3 & 1 \\ -1 & 9 & 6 \\ 0 & 27 & 27 \end{vmatrix}}_{=0} - 1 \begin{vmatrix} 1 & 3 & 1 \\ 0 & 9 & 6 \\ -1 & 27 & 27 \end{vmatrix} + 1 \begin{vmatrix} 1 & 0 & 1 \\ 0 & -1 & 6 \\ -1 & 0 & 27 \end{vmatrix} \underbrace{- 0 \begin{vmatrix} 1 & 0 & 3 \\ 0 & -1 & 9 \\ -1 & 0 & 27 \end{vmatrix}}_{=0}$$

$$= -1\left(1\begin{vmatrix} 9 & 6 \\ 27 & 27 \end{vmatrix} - 3\begin{vmatrix} 0 & 6 \\ -1 & 27 \end{vmatrix} + 1\begin{vmatrix} 0 & 9 \\ -1 & 27 \end{vmatrix}\right) + 1\left(1\begin{vmatrix} -1 & 6 \\ 0 & 27 \end{vmatrix} - 0\begin{vmatrix} 0 & 6 \\ -1 & 27 \end{vmatrix} + 1\begin{vmatrix} 0 & -1 \\ -1 & 0 \end{vmatrix}\right)$$

$$= -1\big(81 - 3(6) + 1(9)\big) + 1\big(-27 + 1(-1)\big)$$

$$= -81 + 9 - 27 - 1 \neq 0$$

$\Rightarrow$ Linearly Independent!

Q2 Additional comments

Many people said $\left\{\begin{array}{l} f_1(x) \text{ \& } f_2(x) \text{ are lin. indep} \\ f_2(x) \text{ \& } f_3(x) \text{ - - - -} \\ f_3(x) \text{ \& } f_4(x) \text{ - - - -} \end{array}\right.$

$\Downarrow$

all 4 are lin. indep.

This is NOT enough!

eg $f_1(x) = x$ $f_2(x) = 1$

$f_3(x) = x + 1$

it is easy to show
f_1 & f_2 indep.
f_2 & f_3 indep
f_1 & f_3 indep

but f_1, f_2, f_3 are not indep.

since

$$1 \cdot f_1 + 1 \cdot f_2 + (-1) \cdot f_3 = 0$$

$$1(x) + 1(1) + (-1)(1+x) = 0$$

Similar phenomena happen for 4 functions
any sub collection of 3 functions may be independent
yet the whole collection of 4 may be dependent!

Q3]... **[25 points]** Find a constant coefficient linear differential operator A such that $Af(x) = 0$ where

$$f(x) = x + e^{2x}.$$

$Dx = 1 \Rightarrow D^2 x = D1 = 0$

$D^2 x = 0$

$(D-2)e^{2x} = 0$

$De^{2x} - 2e^{2x} = 2e^{2x} - 2e^{2x} = 0$

$$\Rightarrow \boxed{\begin{array}{l} D^2(D-2)f(x) = 0 \\ \text{i.e. } A = D^2(D-2) \end{array}}$$

Find the general solution to the following ODE.

$$y^{(2)} - 9y = x + e^{2x}.$$

Associated homogeneous eq^{n}

$(D^2 - 9)y = 0$

$r^2 - 9 = 0$ Char. eq^{n}.

$r^2 = 9$

$r = \pm 3$

e^{3x}, e^{-3x}

$$\boxed{y_h = C_1 e^{3x} + C_2 e^{-3x}}$$

Particular sol^{n}, y_p.

Use Method of undet. coeffs.

$(D^2 - 9)y_p = f(x)$

$\Rightarrow D^2(D-2)(D^2-9)y_p = D^2(D-2)f(x) = 0$

from 1st part!

$$\boxed{D^2(D-2)(D^2-9)y_p = 0}$$

$1, x, e^{2x}, \underbrace{e^{3x}, e^{-3x}}$

ignore these.

$y_p = d_1 + d_2 x + d_3 e^{2x}$

Need to determine these coefficients

$y_p' = d_2 + 2d_3 e^{2x}$ & $y_p'' = 4d_3 e^{2x}$

Thus... $y_p'' - 9y_p = f(x)$ becomes $\underline{4d_3 e^{2x} - 9(d_1 + d_2 x + d_3 e^{2x}) = x + e^{2x}}$

Comparing coeffs on both sides gives...

$$\left\{\begin{array}{lll} -9d_1 = 0 & \Rightarrow & d_1 = 0 \\ -9d_2 = 1 & \Rightarrow & d_2 = -\frac{1}{9} \\ -5d_3 = 1 & \Rightarrow & d_3 = -\frac{1}{5} \end{array}\right\} \Rightarrow y_p = -\frac{x}{9} - \frac{1}{5}e^{2x}$$

General solution is

$$y = y_p + y_h = -\frac{x}{9} - \frac{1}{5}e^{2x} + C_1 e^{3x} + C_2 e^{-3x}$$

Comments on Q3 & Q4

1. Variation of params method will work, but the method of undetermined coeffs is more straightforward.

2. The method of undet. coeffs is NOT a guessing game.

eg. you don't just guess a solⁿ

$$y_p = Ax + Be^{2x} \qquad \text{for Q3}$$

This may work, but sometimes it may lead you astray.

Be systematic ...

step ① Find A so that $A\,f(x) = 0$

eg. $D^2(D-2)\,f(x) = 0$ in Q3.

step ② We want y_p so that $y_p'' - 9y_p = f(x)$

$$(D^2-9)\,y_p = f(x)$$

Apply $D^2(D-2)$ to both sides

$$D^2(D-2)(D^2-9)\,y_p = D^2(D-2)\,f(x) = 0$$

$\Rightarrow y_p$ is a solution of $\boxed{D^2(D-2)(D^2-9)\,y_p = 0}$

$\Rightarrow$ is linear combination of $\underbrace{1, x, e^{2x}}$, $\underbrace{e^{3x}, e^{-3x}}_{\text{ignore these 2.}}$

Step ③ determine the coefficients of these 3.

Q4]... **[25 points]** Find a constant coefficient linear differential operator A such that $Af(x) = 0$ where

$$f(x) = \sin(2x).$$

$f' = 2\cos 2x$

$f'' = -4\sin 2x = -4f$

$f'' + 4f = 0 \Rightarrow \boxed{(D^2+4)f = 0 \qquad A = (D^2+4)}$

Find the general solution to the following ODE.

$$y^{(2)} + 4y = \sin(2x).$$

Associated homogeneous Eq$^{\underline{n}}$

$y^{(2)} + 4y = 0$

$(D^2+4)y = 0$

$\Rightarrow r^2 + 4 = 0$ char. eq$^{\underline{n}}$.

$\Rightarrow r^2 = -4$

$\Rightarrow r = \pm 2i$

$e^{2ix} = \cos 2x + i\sin 2x$

$y_h = C_1 \cos 2x + C_2 \sin 2x$

Find particular sol$^{\underline{n}}$, y_p.

Use Undetermined Coeffs Method

$(D^2+4)y_p = \sin(2x)$ from 1st part

$\Rightarrow (D^2+4)(D^2+4)y_p = (D^2+4)\sin(2x) = 0$

Thus y_p is also a solution of

$(D^2+4)^2 y_p = 0$

$\pm 2i$ with multiplicity 2

$\underbrace{\cos(2x), \sin(2x)}_{\text{Ignore}}, x\cos(2x), x\sin(2x)$

y_p is a combination of these

$y_p = d_1 x\cos 2x + d_2 x \sin 2x$

$y_p' = d_1 \cos 2x - 2d_1 x\sin 2x + d_2 \sin 2x + 2d_2 x\cos 2x$

$y_p'' = -4d_1\sin(2x) - 4d_1 x\cos(2x) + 4d_2\cos(2x) - 4d_2 x\sin(2x)$

Thus $y_p'' + 4y_p = \sin(2x)$ becomes

$-4d_1\sin(2x) - 4d_2\cos(2x) = 1.\sin(2x) + 0.\cos(2x)$

$\Rightarrow d_2 = 0$ & $-4d_1 = 1 \Rightarrow \boxed{d_1 = -\frac{1}{4}}$

$\Rightarrow y_p = -\frac{1}{4}x\cos(2x)$

General sol$^{\underline{n}}$ to ODE

$$y = y_h + y_p = C_1\cos(2x) + C_2\sin(2x) - \frac{1}{4}x\cos(2x)$$