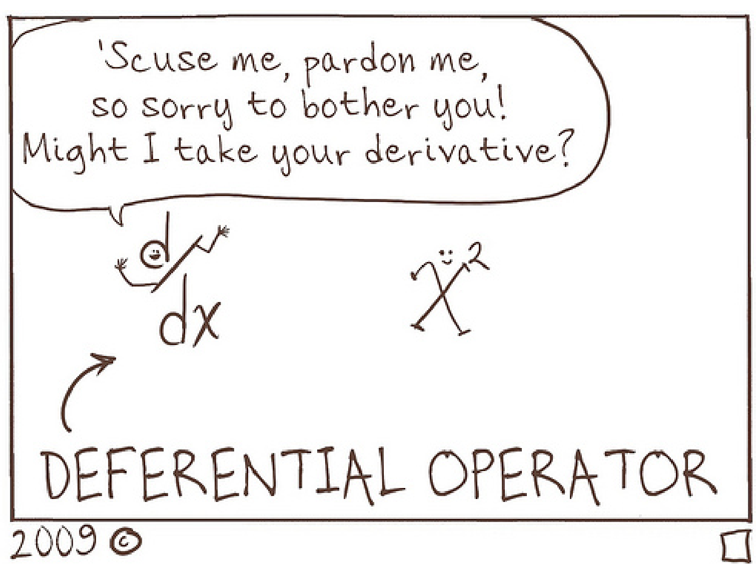
Friday 03/11/2016 Midterm II 50 minutes

Name: Student ID:

Instructions.

1. Attempt all questions.
2. Do not write on back of exam sheets. Extra paper is available if you need it.
3. **Show all the steps of your work clearly. A final answer on its own is not enough for full points.**

Question	Points	Your Score
Q1	25	
Q2	25	
Q3	25	
Q4	25	
TOTAL	100	



Miscellaneous expressions and definitions.

1. **Trig Addition, Half Angle.**

   $\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$ $\qquad \cos(2A) = \cos^2(A) - \sin^2(A)$
   $\cos(2A) = 2\cos^2(A) - 1$ $\qquad \cos(2A) = 1 - 2\sin^2(A)$ $\qquad \sin^2(x) = (1 - \cos(2x))/2$
   $\cos^2(x) = (1 + \cos(2x))/2$ $\qquad \sin(2x) = 2\sin(x)\cos(x)$
   $\sin(A \pm B) = \sin(A)\cos(B) \pm \cos(A)\sin(B)$

2. **Hyperbolic.**

   $\sinh(x) = \frac{1}{2}(e^x - e^{-x})$ $\qquad \cosh(x) = \frac{1}{2}(e^x + e^{-x})$

3. **Integration by Parts.**

   $\int u\,dv \;=\; uv \;-\; \int v\,du$

4. **Integration by substitution.**

$$\int f(u(x))\frac{du}{dx}\,dx \;=\; \int f(u)\,du$$

5. **Inverse Trig.**

   $\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$ $\qquad \frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$ $\qquad \int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}(\frac{x}{a})$

6. **Trig Substitutions.**

   For $\sqrt{a^2 - x^2}$ use $x = a\sin(\theta)$
   For $\sqrt{a^2 + x^2}$ use $x = a\tan(\theta)$
   For $\sqrt{x^2 - a^2}$ use $x = a\sec(\theta)$

7. **Some integrals.**

$$\int \frac{dx}{x} \;=\; \ln|x| + C \qquad\qquad \int \tan(x)\,dx \;=\; \ln|\sec(x)| + C$$

$$\int \sec(x)\,dx \;=\; \ln|\sec(x) + \tan(x)| + C$$

8. **First order linear ODE** $y' + p(x)y = q(x)$ can be solved by first multiplying across by an integrating factor

$$I \;=\; e^{\int p\,dx}$$

9. The equation $M(x,y)dx + N(x,y)dy = 0$ is said to be **exact** if $M_y = N_x$. If it is exact, it can be solved by antidifferentiating M with respect to x and N with respect to y to obtain $F(x,y)$ and then setting $F(x,y) = C$.

10. An ODE of the form $y' = f(ax + by + c)$ can be solved by first making a substitution $v = ax + by + c$.

11. An ODE of the form $y' = f(y/x)$ can be solved by first making a substitution $v = y/x$.

12. The **Bernoulli equation** $y' + p(x)y = q(x)y^n$ can be solved by first making a substitution $v = \frac{1}{y^{n-1}}$.

13. Some **second order ODEs** can be solved by making the substitution $v = y'$.

14. A linear ODE is of the form
$$Ly = f(x) \qquad\qquad (*)$$
where L is a linear differential operator
$$L = y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y.$$
The solution to $(*)$ can be written as
$$y = y_h + y_p$$
where y_p is a particular solution to $(*)$ and y_h is the solution to the associated homogeneous equation
$$Ly = 0 \qquad\qquad (+)$$

15. The general solution of the homogeneous equation is of the form
$$y_h = c_1y_1 + \cdots + c_ny_n$$
where $y_1, \ldots, y_n$ are linearly independent solutions of $(+)$.

16. In the case the a_i are constant functions the solution of $(+)$ is a sum of exponential terms e^{rx} and polynomial times exponential $x^m e^{rx}$ where the r are solutions of the characteristic equation
$$r^n + a_1r^{n-1} + \cdots + a_{n-1}r + a_n = 0.$$
The polynomial times exponential terms occur when the root r repeats with multiplicity greater than 1. Euler's identity
$$e^{(a+ib)x} = e^{ax}(\cos(bx) + i\sin(bx))$$
is useful for dealing with the case when the roots are complex numbers.

17. Particular solutions to $(*)$ can be found by the method of undetermined coefficients. Suppose that there is a linear differential operator A such that $Af(x) = 0$. Then one can look for a particular solution to $(*)$ among the solutions to the homogeneous equation
$$ALy = 0.$$
by finding a suitable linear combination (i.e. determining suitable coefficients) of the linearly independent solutions to $ALy = 0$.

18. Particular solutions to $(*)$ can also be found by the method of variation of parameters. There is a particular solution of the form
$$y_p = c_1y_1 + \cdots + c_ny_n$$
where the y_i are the linearly independent solutions to $(+)$ and the coefficients c_i are now functions of x. One obtains a system of simultaneous equations in $c_1' \ldots, c_n'$ whose coefficient matrix has determinant equal to the Wronskian $W(y_1, \ldots, y_n)$.

19. Simple mechanical vibration systems consisting of a mass m attached to a spring (with constant k) in the presence of friction (with constant c) and an external driving force $f(t)$ give rise to a second order linear ODE
$$mx'' + cx' + kx = f(t).$$

Q1]. . . [25 points] Consider the following ODE.

$$(D^2 - 4D + 5)(D^2 + 4)^2(D - 3)^3 D^4 y \;=\; 0$$

What is its order?

How many independent solutions should it have?

Find the general solution to the ODE above.

Q2]. . . [25 points] Show that the following four functions are linearly independent on $(-\infty, \infty)$.

$$\sin(x), \qquad \cos(x), \qquad e^{3x}, \qquad xe^{3x}$$

Q3]... [25 points] Find a constant coefficient linear differential operator A such that $Af(x) = 0$ where

$$f(x) \;=\; x + e^{2x}.$$

Find the general solution to the following ODE.

$$y^{(2)} - 9y \;=\; x + e^{2x}.$$

Q4]. . . [25 points] Find a constant coefficient linear differential operator A such that $Af(x) = 0$ where

$$f(x) = \sin(2x).$$

Find the general solution to the following ODE.

$$y^{(2)} + 4y = \sin(2x).$$