Mid I - solutions

Q1]...[20 points] Find the general solution to the ODE.

$$\frac{dy}{dx} = ye^x$$

SEPARABLE ODE

$$\Rightarrow \int \frac{dy}{y} = \int e^x\,dx$$

$$\Rightarrow \ln y = e^x + c$$

$$\Rightarrow y = e^{e^x + c} = e^c e^{e^x} = Ae^{e^x}$$

$$\boxed{A = e^c = \text{constant}}$$

General Sol$^{\underline{n}}$ to ODE $\boxed{y = Ae^{e^x}}$ ... $A = $ const.

Find the particular solution to the IVP.

$$\frac{dy}{dx} = ye^x \qquad y(0) = 3e$$

$$3e = y(0) = Ae^{e^0} = Ae^1 = Ae$$

$$\Rightarrow 3 = A$$

$\Rightarrow$ solution to IVP is

$$\boxed{y = 3e^{e^x}}$$

Q2]... **[20 points]** Find the general solution to the ODE.

$$x\frac{dy}{dx} + 2y \;=\; x^2 + 1$$

$P = \frac{2}{x}$ $\quad$ $q = x + \frac{1}{x}$

$$x\frac{dy}{dx} + 2y = x^2 + 1$$

$$\frac{dy}{dx} + \frac{2}{x}y = x + \frac{1}{x}$$ ...of the form $y' + P(x)y = q(x)$

First Order Linear

Multiply across by $I = e^{\int P dx} = e^{\int \frac{2}{x} dx} = e^{2\ln|x|}$

$$= e^{\ln(x^2)} = x^2$$

$$x^2\frac{dy}{dx} + 2xy = x^3 + x$$

$$\frac{d}{dx}(x^2 y) = x^3 + x$$

$$x^2 y = \int x^3 + x \, dx$$

$$= \frac{x^4}{4} + \frac{x^2}{2} + C$$

General Sol. to ODE ...

$$\boxed{y = \frac{x^2}{4} + \frac{1}{2} + \frac{C}{x^2}}$$

... C a constant

Q3]... [20 points] Find the general solution to the following ODE.

$$\frac{dy}{dx} = (x+y+4)^2$$

<u>Substitute</u> $\quad v = x+y+4$

$$\frac{dv}{dx} = 1 + \frac{dy}{dx} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{dv}{dx} - 1$$

$$\frac{dv}{dx} - 1 = v^2$$

$$\frac{dv}{dx} = 1 + v^2 \quad \leftarrow \quad \text{Separable!}$$

$$\Rightarrow \int \frac{dv}{1+v^2} = \int dx$$

$$\Rightarrow \tan^{-1}(v) = x + c$$

$$v = \tan(x+c)$$

$$\Rightarrow x + y + 4 = \tan(x+c)$$

$$\Rightarrow \boxed{y = \tan(x+c) - x - 4} \quad \text{--- } c \text{ a constant}$$

General sol. to ODE

Q4]... **[20 points]** Determine if the following ODE is exact.

$$(6x - y + 3)dx + (4y - x)dy = 0$$

$$\left.\begin{aligned} M_y &= \frac{\partial}{\partial y}(6x - y + 3) = -1 \\ N_x &= \frac{\partial}{\partial x}(4y - x) = -1 \end{aligned}\right\} \Rightarrow M_y = N_x \Rightarrow \underline{\text{exact}}$$

Find the general solution to the ODE above.

$$F(x,y) = \int (6x - y + 3)\,dx$$

$$= 3x^2 - xy + 3x + f(y)$$

$$F(x,y) = \int (4y - x)\,dy$$

$$= 2y^2 - xy + g(x)$$

Comparing gives $g(x) = 3x^2 + 3x$ and $f(y) = 2y^2$. Thus

$$F(x,y) = 3x^2 + 3x + 2y^2 - xy.$$

Solutions to ODE are

$$\boxed{3x^2 + 3x + 2y^2 - xy = C}$$

General solⁿ to ODE

Q5]... **[20 points]** Newton's Law of motion and Hooke's Law of the spring combine to give the following equation describing the motion of a mass m under a spring:

$$m\frac{d^2x}{dt^2} = -kx$$

Here t represents time, the motion is along the x–axis, the spring is in equilibrium (neither stretched nor compressed) at $x = 0$, and k is a positive constant that depends on the particular spring.

By making an appropriate substitution, derive the *law of conservation of energy* for this motion.

Make the substitution

$$\boxed{v = \frac{dx}{dt}}$$

Therefore,

$$\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{dv}{dt} \underset{\text{Ch-Rule}}{=} \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx}\cdot v$$

ODE becomes (after substitution)

$$mv\frac{dv}{dx} = -kx \qquad \leftarrow \text{SEPARABLE ODE}$$

$$\int mv\,dv = \int -kx\,dx$$

$$\frac{mv^2}{2} = \frac{-kx^2}{2} + C \qquad C = \text{constant}.$$

$$\boxed{\tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2 = C} \quad \ldots C \text{ a constant}$$

Kinetic energy of mass + spring potential energy = constant (conserved).