Introduction to Linear Algebra

Linear algebra is the algebra of vectors. In a course on linear algebra you will also learn about the machinery (matrices and reduction of matrices) for solving systems of linear equations. These two subjects are intimately related. We will start off gently recalling facts about vectors from your calculus or physics classes.

Recap. You may recall that a vector has both **magnitude** and **direction.** For example, the position of an object in space (relative to a fixed origin or reference point), the velocity and the acceleration of an object in space are all examples of vectors. Vectors can be represented by directed line segments in the plane (2-d vectors) or in space (3-d vectors). The line segment comes equipped with a little arrow at one end which specifies the direction of the vector, and the length of the line segment specifies the magnitude of the vector. Any two line segments which are of the same length and point in the same direction (in particular the segments are parallel) represent the same vector.

There is another way to represent vectors which is much more useful, and which is much more amenable to generalization. Think about locating a directed line segment in the plane $\mathbb{R}^2$ or in 3-space $\mathbb{R}^3$ so that its starting point is at the origin. Then the line segment (and its direction) is completely determined by the coordinates of the other endpoint. Thus we can use an ordered pair of numbers to represent a 2-d vector (and an ordered triple of numbers to represent a 3-d vector). Here are some examples

$$\langle 1, 1\rangle \qquad\qquad \langle 1, 2, 3\rangle$$

This angle-bracket notation for vectors should be familiar from your calculus book. This way of representing vectors can very easily generalize to 4-d (e.g., $\langle 2, 3, 1, 5\rangle$) or to n-d (e.g., $\langle a_1, \ldots, a_n\rangle$) where our geometric intuition has a much harder time.

Algebra of vectors. There are two fundamental algebraic operations one can perform with vectors. One can **add** two vectors and one can **multiply a vector by a scalar**.

The sum of two vectors is achieved by constructing a triangle (or a parallelogram) when the vectors are given to you as directed line segments. In coordinates, the sum is very easy to describe. For example in 3-d the sum of two vectors is given by

$$\langle x, y, z\rangle + \langle a, b, c\rangle \ = \ \langle x + a, y + b, z + c\rangle$$

and this generalizes nicely to arbitrary dimensions

$$\langle a_1, \ldots, a_n\rangle + \langle b_1, \ldots, n_n\rangle \ = \ \langle a_1 + b_1, \ldots, a_n + b_n\rangle$$

The scalar multiple of a directed line segment is obtained by stretching the line segment by the scalar factor (if the scalar is positive), or scaling the line segment by the absolute value of the scalar and then reversing the direction (if the scalar is negative). In coordinates scalar multiplication is easy to describe

$$c\langle x, y, z\rangle \ = \ \langle cx, cy, cz\rangle$$

and more generally

$$c\langle a_1, \ldots, a_n\rangle \ = \ \langle ca_1, \ldots, ca_n\rangle$$

The collection $\mathbb{R}^2$ of all 2-d vectors is an example of a **vector space**. Another example is $\mathbb{R}^3$ the space of all 3-d vectors, and so is $\mathbb{R}^n$ the space of all n-d vectors. Note that $\mathbb{R}$ plays two roles here: is the vector space of all 1-d vectors, it is also the collection of all scalars.

Linear transformations. Suppose we are given two vector spaces V and W. A function

$$L : V \to W$$

is said to be a *linear transformation* (or *linear mapping*) if it respects addition and scalar multiplication. That is, if v_1 and v_2 are any vectors in V and if $c \in \mathbb{R}$ is a scalar, then

$$L(v_1 + v_2) \;=\; L(v_1) + L(v_2)$$

and

$$L(cv_1) \;=\; cL(v_1)$$

Note that the addition and scalar multiplication on the RHS is in the vector space W.

Let's build an intuition about linear transformations by considering examples.

1. A linear transformation $L_1 : \mathbb{R} \to \mathbb{R}$ is a very special type of function. Indeed, for any $x \in \mathbb{R}$ we have

$$L_1(x) \;=\; L_1(x1) \;=\; xL_1(1)$$

   Here we write the vector x as being a scalar multiple of the vector 1 (where the scalar is x) and remember that L_1 respects scalar multiplication. Note that this equation tells us that the value $L_1(1)$ completely determines all the other values of L_1. But $L_1(1)$ is just some number, call it a. Then the function L_1 is of the form $L_1(x) = ax$. This is a straight line function.

2. A linear transformation $L_2 : \mathbb{R}^2 \to \mathbb{R}$ is a very special type of function. Indeed, for any $\langle x, y\rangle \in \mathbb{R}$ we have

$$L_2(\langle x, y\rangle) \;=\; L_2(x\langle 1,0\rangle + y\langle 0,1\rangle) \;=\; xL_2(\langle 1,0\rangle) + yL_2(\langle 0,1\rangle)$$

   There are two things going on here. First, every vector in $\mathbb{R}^2$ is a sum of scalar multiples of two special vectors $\langle 1,0\rangle$ and $\langle 0,1\rangle$. In your calculus book you have seen these vectors denoted by $\mathbf{i} = \langle 1,0\rangle$ and $\mathbf{j} = \langle 0,1\rangle$, and seen them called unit basis vectors for $\mathbb{R}^2$. Second, the linearity of L_2 implies that the output $L_2(\langle x, y\rangle)$ is equal to a sum of multiples of the two particular outputs $L_2(\langle 1,0\rangle)$ and $L_2(\langle 0,1\rangle)$. These are just numbers, denote them by a and b respectively. This means

$$L_2(\langle x, y\rangle) \;=\; ax + by$$

3. A similar computation using the basis vectors $\mathbf{i} = \langle 1,0,0\rangle$, $\mathbf{j} = \langle 0,1,0\rangle$, and $\mathbf{k} = \langle 0,0,1\rangle$ tells us that a linear function $L_3 : \mathbb{R}^3 \to \mathbb{R}$ is of the form

$$L_3(\langle x, y, z\rangle) \;=\; ax + by + cz$$

   where the numbers a, b, c are just the outputs of L_3 on the basis vectors: $a = L_3(\mathbf{i})$, $b = L_3(\mathbf{j})$, and $c = L_3(\mathbf{k})$.

Comments.

1. You might want to think about and write down the form of a linear function $L : \mathbb{R}^n \to \mathbb{R}$.

2. You can drop the $\langle\rangle$ brackets and write these functions more like calc 3 functions. $L_1(x) = ax$, $L_2(x, y) = ax + by$, and $L_3(x, y, z) = ax + by + cz$.

3. Show that a general linear function $L : \mathbb{R}^n \to \mathbb{R}$ is of the form $L(x_1, \ldots, x_n) = a_1x_1 + \cdots + a_nx_n$ where the a_i are real numbers, and a_i is the output of L on the basis vector $\langle 0, \ldots, 0, 1, 0, \ldots, 0\rangle$ which has a 1 in the ith position and 0's elsewhere.

Linear Equations. Suppose that $L : \mathbb{R}^n \to \mathbb{R}$ is a linear transformation. Let us think about the solutions to linear equations of the form

$$L(\mathbf{v}) = d$$

where $d \in \mathbb{R}$ and $\mathbf{v} \in \mathbb{R}^n$ is an n–tuple of real numbers. Start with a few examples.

1. If $L(x, y) = ax + by$ then $L(x, y) = d$ becomes the equation $ax + by = d$. We know that the set of solutions is a straight line in the plane, $\mathbb{R}^2$.

2. If $L(x, y, z) = ax + by + cz$ then $L(x, y, z) = d$ becomes the equation $ax + by + cz = d$. We know that the set of solutions is a plane in 3-d space, $\mathbb{R}^3$.

We have a good intuition about lines and planes, and so have a good intuition about these solution sets. We might even convince ourselves that solutions to the equation $L(x_1, \dots, x_n) = d$ is some type of $(n-1)$–dimensional hyperplane in $\mathbb{R}^n$.

However, it is worthwhile thinking about these solution sets in another way. A way which is reminiscent of the parametric descriptions of lines and planes that you may recall from your vector calculus class. Let's revisit the two examples above in detail.

1. Examples of the form $ax + by = d$. For concreteness take $L(x, y) = 2x + 3y$ and $d = 3$, so that the equation $L(x, y) = d$ becomes

   $$2x + 3y = 3 \qquad (*)$$

   We know from high school algebra that the set of solutions forms a line in the plane. Let us think about the solution set in another way which is closely related to the parametric description of a line.

   If (x_1, y_1) and (x_2, y_2) are two solutions to $(*)$ then, since L is linear,

   $$L((x_1, y_1) - (x_2, y_2)) = L(x_1, y_1) - L(x_2, y_2) = 3 - 3 = 0$$

   That is the difference between any two solutions of $(*)$ is a solution of the *associated homogeneous equation*

   $$2x + 3y = 0 \qquad (**)$$

   This equation is called homogeneous because if (x, y) is a solution then so is any multiple (cx, cy). The general solution to $(**)$ is easier to describe. The equation implies that $y = -2x/3$, so the most general solution is $(x, -2x/3)$ for any real number x. Let's clear the 3 in the denominator, and write c in place of x. The general solution of $(**)$ is the collection of all 2-d vectors of the form

   $$c(3, -2) \qquad \text{where } c \in \mathbb{R}.$$

   Finally, we can see by substitution that $(0, 1)$ is a solution to $(*)$. If (x, y) is a general solution of $(*)$ then $(x, y) - (0, 1)$ is a general solution of $(**)$. Thus

   $$(x, y) - (0, 1) = c(3, -2) \qquad \text{where } c \in \mathbb{R}.$$

   or rewriting

   $$(x, y) = (0, 1) + c(3, -2) \qquad \text{where } c \in \mathbb{R}. \qquad (+)$$

   This agrees with our geometric intuition about the line $2x + 3y = 3$. In order to get to a general point on this line, you can go to the point $(0, 1)$ which lies on the line, and then travel a multiple (c times in this case) of a parallel vector to the line. This description of points on the line $2x + 3y = 3$ is called a parametric description of the line. Here c is the parameter.

   Algebraically, the general solution $(+)$ has a nice description. *The general solution of equation $(*)$ is the sum of a particular solution of $(*)$ and the general solution of the associated homogenous equation $(**)$.*

2. Here is a second example. Let $L(x, y, z) = 2x + 3y + z$ be a linear map from $\mathbb{R}^3$ to $\mathbb{R}$. The equation

$$L(x, y, z) \;=\; 6 \qquad\qquad (*)$$

has a solution set which we recognize from our calculus course to be the plane $2x + 3y + z = 6$ in 3-d space. However, we can also work using linearity to obtain another description of the solution set.

As above, if (x_1, y_1, z_1) and (x_2, y_2, z_2) are two solutions to $(*)$ then their difference $(x_1, y_1, z_1) - (x_2, y_2, z_2)$ satisfies (by linearity of L)

$$L((x_1, y_1, z_1) - (x_2, y_2, z_2)) \;=\; L(x_1, y_1, z_1) - L(x_2, y_2, z_2) \;=\; 6 - 6 \;=\; 0$$

That is, the difference of two solutions of $(*)$ is a solution of the associated homogeneous equation

$$2x + 3y + z \;=\; 0 \qquad\qquad (**)$$

This homogeneous equation is easier to deal with. By inspection we can read off two particular solutions of $(**)$; for example, $(1, 0, -2)$ and $(0, 1, -3)$. If (x, y, z) is a general solution of $(**)$ then we know that $2x + 3y + z = 0$ or, in other words, $z = -2x - 3y$. Then our general solution becomes

$$(x, y, -2x - 3y) \;=\; (x, 0, -2x) + (0, y, -3y) \;=\; x(1, 0, -2) + y(0, 1, -3)$$

Thus the general solution of $(**)$ can be expressed in a 2-parameter fashion

$$c_1(1, 0, -2) + c_2(0, 1, -3)$$

where $c_1, c_2 \in \mathbb{R}$.

Noe by inspection that $(1, 1, 1)$ is a particular solution to equation $(*)$.

Finally, arguing as in example 1 we get the following. *The general solution of* $(*)$ *is the sum of a particular solution to* $(*)$ *and the general solution of the associated homogeneous equation* $(**)$.

$$(x, y, z) \;=\; (1, 1, 1) + c_1(1, 0, -2) + c_2(0, 1, -3)$$

As before, there is a nice geometric interpretation. In order to get to a general point on the plane $2x + 3y + z = 6$ one first gets to the point $(1, 1, 1)$ in this plane, and then travels along multiples of two independent (i.e., not parallel to each other) vectors lying parallel to this plane. This is called the parametric description of the plane. In your calculus class you may have used the variables s and t for parameters instead of c_1 and c_2.

3. Do a similar analysis for the n-dimensional case. Argue that the general solution to a linear equation of the form

$$a_1x_1 + \cdots + a_nx_n \;=\; b \qquad\qquad (*)$$

is the sum of a particular solution $(p_1, \ldots, p_n)$ to $(*)$, and the general solution to the associated homogeneous equation.

$$a_1x_1 + \cdots + a_nx_n \;=\; 0 \qquad\qquad (**)$$

In turn, you can argue that the most general solution to $(**)$ is a sum

$$c_1\mathbf{v}_1 + \cdots + c_{n-1}\mathbf{v}_{n-1}$$

where

$$\mathbf{v}_i \;=\; (0, \ldots, 0, -a_n, 0, \ldots, 0, a_i)$$

here we have $-a_n$ in position i and a_i in position n and zeros elsewhere. We are assuming that $a_n \neq 0$ here. Think about what you might use if $a_n = 0$.

In summary, the general solution of $(*)$ has an $(n-1)$ parameter description

$$(x_1, \ldots, x_n) \;=\; (p_1, \ldots, p_n) + c_1\mathbf{v}_1 + \cdots + c_{n-1}\mathbf{v}_{n-1}$$

where the parameters $c_1, \ldots, c_{n-1} \in \mathbb{R}$ and where the $\mathbf{v}_i$ are defined above.

The message to take away. If $L : \mathbb{R}^n \to \mathbb{R}$ is a linear transformation, then the general solution of a linear equation

$$L(\mathbf{v}) \;=\; w$$

can be written as

$$\mathbf{v} \;=\; \mathbf{v}_0 + c_1\mathbf{v}_1 + \cdots + c_{n-1}\mathbf{v}_{n-1}$$

where

- $\mathbf{v}_0$ is a particular solution to $(*)$
- $c_1, \ldots, c_{n-1} \in \mathbb{R}$ are arbitrary parameters, and
- $\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}$ are solutions of the associated homogeneous equation $L(\mathbf{v}) = 0$.

It is important to note that what makes all this work is the fact that L is a linear transformation which respects addition and scalar multiplication. We will use these facts again in a setting which *a priori* looks nothing like the above.

The ODE setting. We know of other objects that can be added and multiplied by scalars; functions $f : \mathbb{R} \to \mathbb{R}$. We also know of a transformation of functions which respects addition and scalar multiplication; differentiation. Let's make this more precise.

The vector spaces $\mathbb{R}^n$ and $\mathbb{R}$ will be replaced by the collection V of all infinitely differentiable functions:

$$V \;=\; \{f \mid f : \mathbb{R} \to \mathbb{R} \text{ is infinitely differentiable }\}$$

and L will be replaced by a *linear differential operator*. The key linear aspects of the previous setup remain in this setting.

- two infinitely differentiable functions can be added to yield a third. The addition is defined pointwise by adding outputs.
$$(f+g)(x) \;=\; f(x) + g(x)$$
- an infinitely differentiable function can be multiplied by a scalar ($c \in \mathbb{R}$) to yield another.
$$(cf)(x) \;=\; c(f(x))$$
- The operation of differentiation
$$D(f) \;=\; \frac{df}{dx}$$
is a *linear transformation* $D : V \to V$. Indeed, it respects sums
$$D(f+g) \;=\; \frac{d(f+g)}{dx} \;=\; \frac{df}{dx} + \frac{dg}{dx} \;=\; D(f) + D(g)$$
(here we are using a result from calc 1), and it respects scalar multiplication
$$D(cf) \;=\; \frac{d(cf)}{dx} \;=\; c\frac{df}{dx} \;=\; cD(f)$$
(another calc 1 result).

You should check that iterations of D, namely D^2, D^3 etc (which denote the second derivative, third derivative etc), are also linear. So also is the operation of multiplication by an infinitely differentiable function $a(x)$

$$a(x)(f+g) \;=\; a(x)f + a(x)g$$

and

$$a(x)(cf) \;=\; ca(x)f \text{ for } c \in \mathbb{R}.$$

Combining the above we obtain the fact that

$$L \;=\; a_n(x)D^n + \cdots + a_1(x)D + a_0(x)$$

is a linear transformation $L : V \to V$.

The ODE

$$Ly \;=\; f(x) \qquad\qquad (*)$$

written explicitly as

$$(a_n(x)D^n + \cdots + a_1(x)D + a_0(x))y \;=\; f(x)$$

which becomes

$$a_n(x)D^n y + \cdots + a_1(x)Dy + a_0(x)y \;=\; f(x) \qquad\qquad (*)$$

is called an nth order linear differential equation. It should not come as much of a surprise to you that the general solution of $(*)$ can be written as

$$y \;=\; y_p + y_h \qquad\qquad (+)$$

where y_p is a particular solution of $(*)$ and y_h is the general solution of the associated homogeneous equation

$$Ly \;=\; 0 \qquad\qquad (**)$$

It might not also be a surprise that the general solution to $(**)$ can be given a description using some number of parameters (n in this case). We will see details in class. For now it is good to see the analogy with simpler linear equations, lines, planes etc.

Exercise. The first order linear ODE

$$y' + p(x)y \;=\; q(x)$$

can be written in the form

$$L(y) \;=\; q(x)$$

where

$$L \;=\; D + p(x)$$

is a linear differential operator. Check that the solution we met in chapter 1, namely

$$y \;=\; e^{-\int p\,dx}\left(\int q(x)e^{\int p\,dx}\,dx + C\right)$$

is really the sum of two functions. One is a particular solution to $L(y) = q(x)$ and the other is the general (one-parameter) solution to the associated homogeneous equation $L(y) = 0$. Woohoo! This example fits into our general linear framework.