Linear Systems of ODEs

1. **Vector valued functions.** Let

$$\mathbf{x} = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

be an $n \times 1$ column vector of functions of t. Recall from your vector calculus class that we define its derivative to be

$$D\mathbf{x} = \frac{d\mathbf{x}}{dt} = \begin{pmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{pmatrix}$$

2. **Linear System.** An *nth order linear system of ODEs* is a system of ODEs of the form

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} + \mathbf{f} \qquad (*)$$

where $\mathbf{x}$ and f are $n \times 1$ column vectors of functions of t and A is an $n \times n$ matrix whose entries are functions of t.

The functions in A and $\mathbf{f}$ are given to you, and you have to find a column vector $\mathbf{x}$ of n functions of t which satisfies $(*)$.

3. **Homogeneous system.** The linear system is said to be *homogeneous* if $\mathbf{f} = \mathbf{0}$. That is

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} \qquad (+)$$

is a homogeneous system. It is called the homogeneous system *associated* to $(*)$.

We can write equation $(+)$ as

$$(D - A)\mathbf{x} = 0$$

so that it looks more like the format of homogeneous linear ODEs from before.

4. **Examples/Applications.** Linear systems of ODEs are very useful. For example the general nth order linear ODE

$$y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_0(t)y = f(t)$$

can be encoded as the linear system

$$\begin{aligned} \frac{dx_1(t)}{dt} &= x_2(t) \\ \frac{dx_2(t)}{dt} &= x_3(t) \\ \vdots \quad & \quad \vdots \\ \frac{dx_{n-1}(t)}{dt} &= x_n(t) \end{aligned}$$

and

$$\frac{dx_n(t)}{dt} = f(t) - a_0(t)x_1 - \cdots - a_{n-1}(t)x_n$$

where $x_1(t) = y, x_2(t) = y', \ldots, x_n(t) = y^{(n-1)}$. In other words,

$$D\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{n-1} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f(t) \end{pmatrix}$$

Also, more complex mass spring systems (involving several coupled masses and springs), complex RLC circuit diagrams (with several loops), and cascading mixing problem setups with feedback pipes can all be encoded using linear systems.

5. **Existence/Uniqueness.** Suppose that the functions $a_{ij}(t)$ and $f_i(t)$ in the linear system $(*)$ above are all continuous on the open interval I, and let a be a point of I. Then given any $\mathbf{b} \in \mathbb{R}^n$ there exists a unique solution $\mathbf{x}$ to $(*)$ defined on I such that

$$\mathbf{x}(a) \;=\; \mathbf{b}.$$

6. **General solution of a linear system.** This theory mirrors the theory for nth order ODEs precisely.

   The general solution to the system $(*)$ is a sum

$$\mathbf{x} \;=\; \mathbf{x}_p + \mathbf{x}_h$$

   where $\mathbf{x}_p$ is a *particular solution* to $(*)$ and $\mathbf{x}_h$ is the general solution to the associated homogeneous equation $(+)$.

   Furthermore, the general solution to the nth order homogeneous equation $(+)$ can be written as a *linear combination*

$$\mathbf{x}_h \;=\; c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n$$

   where the c_i are real numbers and the $\mathbf{x}_i$ form a *linearly independent* set of solutions to $(+)$.

7. **Linear independence.** The set of solutions $\mathbf{x}_1, \ldots, \mathbf{x}_n$ to $(+)$ is *linearly independent* if the only linear combination

$$c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n \;=\; \mathbf{0}$$

   yielding the zero vector is the trivial combination $c_1 = 0, \ldots, c_n = 0$.

   Equivalently, the solutions $\mathbf{x}_1, \ldots, \mathbf{x}_n$ to $(+)$ form a linearly independent set if the Wronskian

$$W(\mathbf{x}_1, \ldots, \mathbf{x}_n) \;\neq\; 0$$

   where the Wronskian is defined to be the determinant of the $n \times n$ matrix with column vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$.

8. **Solving linear systems of ODEs by elimination.** From now on we focus on linear systems where the coefficient matrix A consists of n^2 constants. The following method can be generalized to work for nth order linear systems. Suppose $L_1, \ldots, L_4$ are constant coefficient linear differential operators and x, y, f_1, f_2 are functions of t. Then the system

$$\begin{aligned} L_1 x + L_2 y &= f_1 \\ L_3 x + L_4 y &= f_2 \end{aligned}$$

can be solved by eliminating y (or x) and considering the following ODEs in x and y alone

$$(L_1L_4 - L_2L_3)x = L_4f_1 - L_2f_2$$

and

$$(L_1L_4 - L_2L_3)y = L_1f_2 - L_3f_1$$

This may be written symbolically as

$$\det\begin{pmatrix} L_1 & L_2 \\ L_3 & L_4 \end{pmatrix} x = \det\begin{pmatrix} f_1 & L_2 \\ f_2 & L_4 \end{pmatrix}$$

and

$$\det\begin{pmatrix} L_1 & L_2 \\ L_3 & L_4 \end{pmatrix} y = \det\begin{pmatrix} L_1 & f_1 \\ L_3 & f_2 \end{pmatrix}$$

9. **Eigenvalues and Eigenvectors.** A (real or possibly complex) number λ is said to be an *eigenvalue* of the $n \times n$ matrix A if there is a non-zero vector $\mathbf{v} \in \mathbb{C}^n$ (that is, a column vector with real or complex entries) such that

$$A\mathbf{v} = \lambda\mathbf{v}$$

The non-zero vector $\mathbf{v}$ is said to be an *eigenvector* of A corresponding to the eigenvalue λ (or a λ–eigenvector).

Note that if $\mathbf{v}$ is a λ–eigenvector of A, then the matrix equation

$$(A - \lambda I)\mathbf{v} = \mathbf{0}$$

has a non-zero solution $\mathbf{v}$. Here I is the $n \times n$ identity matrix. This equation has a non-zero solution when the determinant of $(A - \lambda I)$ is zero.

Let's summarize the procedure for finding eigenvalues and eigenvectors of A.

- First, one finds eigenvalues of A by solving the *characteristic equation*

$$\det(A - \lambda I) = 0.$$

- Next, for each eigenvalue λ, one finds λ–eigenvectors by finding non-zero solutions to the equation

$$(A - \lambda I)\mathbf{v} = \mathbf{0}.$$

10. **Eigenvalue/Eigenvector Problems.** It is good to have an understanding of matrices and linear transformations from the eigenvalue/eigenvector perspective. We will see why this is useful for systems of ODEs shortly. But eigenvectors and eigenvalues are important in many applications of mathematics to physics, engineering, biology, statistics and data analysis. The *Google page-rank algorithm* (the theoretical framework behind the very successful Google search engine and a multibillion dollar enterprise) is an algorithm for solving a huge eigenvalue-eigenvector problem.

11. **Solving homogeneous linear systems of ODEs by the eigenvalue method.** Let A be an $n \times n$ matrix with constant entries. Suppose that λ is an eigenvalue of A and that $\mathbf{v}$ is a λ–eigenvector. Then the homogeneous system

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} \qquad (+)$$

has a solution

$$\mathbf{x}(t) = e^{\lambda t}\mathbf{v} \qquad (\#)$$

This result is fundamental. This means that in order to find a general solution to (+) we should

- Find the eigenvalues of A.
- For each eigenvalue λ above, find the largest collection of linearly independent λ–eigenvectors.
- If we obtain a linearly independent collection of n eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ above and form the corresponding solutions $\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)$ using the formula in (#), then

  $$\mathbf{x}_h(t) \;=\; c_1\mathbf{x}_1(t) + \cdots + c_n\mathbf{x}_n(t) \;=\; c_1 e^{\lambda_1 t}\mathbf{v}_1 + \cdots + c_n e^{\lambda_n t}\mathbf{v}_n$$

  is the general solution to (+).

Sounds cool. And it is cool when it works! There are a few issues to deal with. We list them here and describe how to deal with them in subsequent numbered items.

- One, we have to be careful interpreting things in the case that an eigenvector λ is not a real number. In this case, the λ–eigenvectors will have complex entries, and we will need to use Euler's identity to obtain expressions for solutions to the original real number coefficient homogeneous system (+).
- Two, we may not obtain a collection of n linearly independent eignevectors. This is a more serious issue. In this case, we will have to work with *generalized eigenvectors* and use formulas which are reminiscent of how we dealt with repeated roots in the case of a linear nth order ODE.

12. **Dealing with a non-real eigenvalue λ.** In the case λ is not a real number you should still proceed as in the real case. That is, you should solve

    $$(A - \lambda I)\mathbf{v} \;=\; 0$$

    to obtain λ–eigenvectors with complex entries. Now, the solution

    $$\mathbf{x}(t) \;=\; e^{\lambda t}\mathbf{v} \;=\; \mathbf{u}(t) + i\mathbf{v}(t)$$

    will decompose as a sum $\mathbf{u}(t) + i\mathbf{v}(t)$ of real and imaginary vectors. Then you use $\mathbf{u}(t)$ and $\mathbf{v}(t)$ as the two real vector solutions to the original linear system. This is formally identical to the way we dealt with complex solutions to the characteristic equation in the case of linear ODEs, the only difference now is that we are working with column vectors.

13. **Working with generalized eigenvalues.** An eigenvector λ is a solution to the characteristic equation.

    $$\det(A - \lambda I) \;=\; 0$$

    As such λ may have multiplicity m. It may happen that you can only find r linearly independent λ–eigenvectors where $r < m$. In this case you should look for $m - r$ *generalized eigenvectors*. Suppose $\mathbf{v}_1$ is a λ–eigenvector. Then generalized λ–eigenvectors $\mathbf{v}_2, \ldots, \mathbf{v}_k$ are found by solving (recursively)

    $$(A - \lambda I)\mathbf{v}_2 \;=\; \mathbf{v}_1\,, \ldots, (A - \lambda I)\mathbf{v}_k \;=\; \mathbf{v}_{k-1}$$

    In this case

    $$\begin{aligned}
    \mathbf{x}_1 &= e^{\lambda t}\mathbf{v}_1 \\
    \mathbf{x}_2 &= e^{\lambda t}(t\mathbf{v}_1 + \mathbf{v}_2) \\
    \mathbf{x}_3 &= e^{\lambda t}(\tfrac{t^2}{2}\mathbf{v}_1 + t\mathbf{v}_2 + \mathbf{v}_3) \\
    &\vdots \\
    \mathbf{x}_k &= e^{\lambda t}(\tfrac{t^{k-1}}{(k-1)!}\mathbf{v}_1 + \cdots + t\mathbf{v}_{k-1} + \mathbf{v}_k)
    \end{aligned}$$

form a chain of k linearly independent solutions to the linear system.

Note that for a given eigenvalue λ, there may be several independent λ–eigenvectors, each giving rise to its own chain of generalized λ–eigenvectors (and these chains may be of different lengths). The key algebraic fact is that the sum of the lengths of all these chains will be equal to m, the multiplicity of the eigenvalue λ.

14. **The matrix exponential.** Let A be an $n \times n$ matrix. We denote the matrix exponential of tA by e^{tA} and define it to be the $n \times n$ matrix given by the series

$$e^{tA} \;=\; I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots$$

Properties.

- $e^{t0} = I$ where 0 is the $n \times n$ matrix with all entries 0 and I is the identity $n \times n$ matrix.
- $e^{tI} = \mathrm{diag}(e^t, \ldots, e^t)$.
- $\frac{d}{dt}e^{tA} = Ae^{tA}$.
- If $AB = BA$, then $e^{tA+tB} = e^{tA}e^{tB}$.
- Suppose that

$$J \;=\; \begin{pmatrix} \lambda & 1 & 0 & \ldots & 0 \\ 0 & \lambda & 1 & & \vdots \\ & & \ddots & \ddots & 0 \\ \vdots & & & \lambda & 1 \\ 0 & \ldots & & 0 & \lambda \end{pmatrix}$$

with λ on the diagonal, 1 just above the diagonal, and 0 everywhere else. Then we can write $J = \lambda I + N$ where $N = J - \lambda I$ has 0 on the diagonal.
In this case $N\lambda I = \lambda I N$ and so

$$e^{tJ} \;=\; e^{t\lambda I + tN} \;=\; e^{t\lambda I}e^{tN} \;=\; e^{t\lambda}e^{tN}$$

where

$$e^{tN} \;=\; \begin{pmatrix} 1 & t & \frac{t^2}{2!} & \ldots & \frac{t^{n-1}}{(n-1)!} \\ 0 & 1 & t & & \vdots \\ & & \ddots & \ddots & \frac{t^2}{2!} \\ \vdots & & & 1 & t \\ 0 & \ldots & & 0 & 1 \end{pmatrix}$$

15. **The matrix exponential and homogeneous linear systems of ODEs.** The homogeneous system

$$\frac{d\mathbf{x}}{dt} \;=\; A\mathbf{x} \qquad\qquad (+)$$

has a solution

$$\mathbf{x}(t) \;=\; e^{tA}\mathbf{x}(0).$$

By choosing $\mathbf{x}(0)$ to be the standard basis vectors, we obtain n linearly independent solutions to $(+)$. Note that these solutions are precisely the column vectors of the matrix exponential e^{tA}.