Linear nth Order ODEs with constant coefficients.

1. The general form of a linear nth order ODE with constant coefficients is

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}y' + a_n y \;=\; f(x) \qquad (*)$$

where the a_i are real numbers and $f(x)$ is continuous on some interval I. This can be written as

$$Ly \;=\; f(x)$$

where L is the linear differential operator

$$L = D^n + a_1 D^{n-1} + \cdots + a_{n-1}D + a_n.$$

2. We will focus first of all on solving the associated homogeneous equation

$$Ly \;=\; 0 \qquad (+)$$

This is achieved by looking for solutions of the form $y = e^{rx}$. Substituting into (+) gives the characteristic equation

$$r^n + a_1 r^{n-1} + \cdots + a_{n-1}r + a_n \;=\; 0. \qquad (\#)$$

3. The characteristic equation is a polynomial equation. You may recall solving certain quadratic and cubic equations in high school by first factoring the polynomial on the left hand side (LHS). There is a theorem called *The Fundamental Theorem of Algebra* which states that the LHS of (#) can always be factored into a product of n linear factors $(r - \alpha)$ provided you allow the possibility that some of the α may be complex numbers. Remember, if $(r - \alpha)$ is a factor, then $r = \alpha$ is a solution to the equation (also known as a *root* of the polynomial).

   A factor $(r - \alpha)$ may occur with some power, such as $(r - \alpha)^m$; in this case we say that the root α repeats with multiplicity m.

4. The cool fact about constant coefficient linear differential operators is that the linear terms commute

$$(D - \alpha)(D - \beta) \;=\; (D - \beta)(D - \alpha)$$

   This means that we can "factor" and "rearrange" nth order constant coefficient linear differential operators L in just the same way that we factored and rearranged degree n polynomials in high school.

   Note that this property crucially relies on the fact that α and β are constants. For example, you should verify that

$$(D - x)D \;\neq\; D(D - x)$$

   by evaluating both sides on y (a function of x).

5. Now we can say what linearly independent functions one can use to form the general solution of (+). Recall that a fundamental set consists of n solutions with the property that any solution can be written as a linear combination of these n.

   - If r is a real root with multiplicity m, then

$$e^{rx}, xe^{rx}, \ldots, x^{m-1}e^{rx}$$

     are m functions in the fundamental set. This is one function e^{rx} in the case $m = 1$.

- If $r + i\omega$ is a complex root with multiplicity m, then its complex conjugate $r - i\omega$ is another root with multiplicity m and

$$e^{rx}\cos(\omega x), e^{rx}\sin(\omega x), xe^{rx}\cos(\omega x), xe^{rx}\sin(\omega x), \ldots, x^{m-1}e^{rx}\cos(\omega x), x^{m-1}e^{rx}\sin(\omega x)$$

  are $2m$ functions in the fundamental set. This is two functions $e^{rx}\cos(\omega x), e^{rx}\sin(\omega x)$ in the case $m = 1$.

This is all there is to say about the general solution of a homogeneous linear nth order ODE with constant coefficients. Notice that it is all polynomial algebra (roots of polynomial equations), and that there is no antiderivatives to compute!

6. **Proof of the facts about fundamental sets.** In the case r is a real root with multiplicity m, then $(D - r)^m$ is a factor of the linear differential operator L. Use the product rule to verify that for any function u

$$(D - r)(e^{rx}u) \ = \ e^{rx}Du$$

Thus, $(D - r)^m(e^{rx}u) = e^{rx}D^m u$ and so $(D - r)^m(e^{rx}u) = 0$ gives the condition $D^m u = 0$. Thus u is a polynomial of degree $m - 1$, and we have proven the first bullet point above. It is an exercise to check that $e^{rx}, xe^{rx}, \ldots, x^{m-1}e^{rx}$ are linearly independent.

This same reasoning holds for complex roots $r \pm i\omega$ of multiplicity m. First find the solutions corresponding to $(D - (r \pm i\omega))y = 0$ and then multiply them by powers of x up through x^{m-1}. From our class notes on complex numbers the solution $e^{(r \pm i\omega)x}$ can be rewritten using Euler's identity as

$$e^{rx}(\cos(\omega x) \pm i\sin(\omega x))$$

and this gives the two independent real solutions $e^{rx}\cos(\omega x)$ and $e^{rx}\sin(\omega x)$.

7. **Solving $(*)$ for non-zero RHS — Undetermined Coefficients.** The following algorithm finds a particular solution to $(*)$ in the case that the RHS function $f(x)$ is a non-zero function of one of the following types:

- a constant function
- an exponential e^{rx}
- a sinusoid $\cos(\omega x)$ or $\sin(\omega x)$
- $e^{rx}\cos(\omega x)$ or $e^{rx}\sin(\omega x)$
- a positive integer power of x times any of the above.

Step 1. Find a constant coefficient linear differential operator A with the property that $A(f) = 0$.

Step 2. Notice that if y_p is a particular solution to $(*)$, then

$$AL(y_p) \ = \ Af \ = \ 0 \qquad\qquad (++)$$

Step 3. Thus y_p can be expressed as a linear combination

$$c_1 y_1 + \cdots + c_N y_N$$

of the fundamental solutions to the homogeneous equation $(++)$ (which has order $N > n$).

Step 4. Rule out multiples of the n fundamental solutions which happen to be solutions of $(+)$ (that is, of $Ly = 0$).

Step 5. Plug the general linear combination of the remaining $(N - n)$ solutions into $(*)$ and determine the coefficients c_j which give a particular solution y_p.

8. **Solving (∗) for non-zero RHS — Variation of Parameters.** This method also produces a particular solution y_p of (∗) in the case that the RHS $f(x)$ is non-zero. It works for more general RHS functions than the Undetermined Coefficients method, although it requires more algebra and some antidifferentiation. Here are the steps.

Step 1. Write down the general solution

$$y = c_1 y_1 + \cdots + c_n y_n$$

of the associated homogeneous equation (+).

Step 2. Replace the parameters c_i by functions $c_i(x)$ (that is, allow the parameters to vary) and consider a solution of (∗) of the form

$$y_p = c_1(x) y_1 + \cdots + c_n(x) y_n$$

Step 3. Solve the equations

$$\begin{aligned} c_1'(x)y_1 + \cdots + c_n'(x)y_n &= 0 \\ c_1'(x)y_1' + \cdots + c_n'(x)y_n' &= 0 \\ \vdots \quad &= \vdots \\ c_1'(x)y_1^{(n-1)} + \cdots + c_n'(x)y_n^{(n-1)} &= f(x) \end{aligned}$$

for the functions $c_i'(x)$. This is possible since the y_i are linearly independent solutions to a homogeneous ODE and so $W(y_1, \ldots, y_n) \neq 0$.

Step 4. Integrate to obtain functions $c_1(x), \ldots, c_n(x)$.

Step 5. A particular solution to (∗) is given by

$$y_p = c_1(x) y_1 + \cdots + c_n(x) y_n.$$