First Order Equations – Methods

Here is a summary of our methods of solving various first order ODEs (and some second order ODEs).

1. **Repeated antidifferentiation.**

$$\text{ODE:} \qquad \frac{dy}{dx} = f(x)$$

$$\text{SOL:} \qquad y = \int f(x)\,dx + C$$

Also works for $y'' = f(x)$ etc. Good for acceleration, velocity, position problems.

2. **Separable ODE.**

$$\text{ODE:} \qquad \frac{dy}{dx} = f(x)g(y)$$

$$\text{SOL:} \qquad \int \frac{dy}{g(y)} = \int f(x)\,dx + C$$

3. **First order linear.**

$$\text{ODE:} \qquad \frac{dy}{dx} + p(x)y = q(x)$$

SOL: Multiply the equation across by $I(x) = e^{\int p(x)dx}$. The LHS will become the output of a product rule: $\frac{d(I(x)y)}{dx}$, and so the equation is solved by antidifferentiating the new RHS, $I(x)q(x)$, and then dividing by $I(x)$.

4. **Exact equations.**

$$\text{ODE:} \qquad M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

SOL: This is exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. In such case it is possible to find a function $F(x,y)$ that satisfies

$$F(x,y) = \int M(x,y)\,dx + f(y)$$

$$F(x,y) = \int N(x,y)\,dy + g(x)$$

Solutions to the ODE are level curves of F

$$F(x,y) = C$$

5. **Substitution Methods.**

$$\text{ODE:} \qquad \frac{dy}{dx} = f(v)$$

for some $v = v(x,y)$.

SOL: Rewrite $\frac{dv}{dx} = \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\frac{dy}{dx}$. Use this to rewrite the equation as an equation in v', v, x.

6. **Bernoulli Equations.**

$$\text{ODE:} \qquad \frac{dy}{dx} + q(x)y \;=\; p(x)y^n$$

for some $n \in \mathbb{R}$.

SOL: Use the substitution $v = 1/y^{(n-1)}$ to rewrite the equation as a linear first order equation in v', v, and x:

$$\frac{dv}{dx} + (1-n)p(x)v \;=\; (1-n)q(x).$$

7. **Second order $\longrightarrow$ First order.**

$$\text{ODE:} \qquad F(x, y', y'') = 0$$

SOL: Use the substitution $v = y'$ to get a first order equation $F(x, v, v') = 0$.

$$\text{ODE:} \qquad F(y, y', y'') = 0$$

where derivatives are **with respect to** x.

SOL: Use the substitution $v = y'$ and $y'' = v' = \frac{dv}{dy}\frac{dy}{dx} = \frac{dv}{dy}v$ to get a first order equation $F(y, v, v\frac{dv}{dy}) = 0$ in y, v and $\frac{dv}{dy}$. From the solution $v = v(y)$ we obtain

$$x \;=\; \int \frac{dx}{dy}\,dy + C \;=\; \int \frac{dy}{v} + C$$

which is a solution to the original equation.