Q1]... **[12 points]** Test to see if the following ODE is exact. If it is exact, then find a solution.

$$\underbrace{(2x^2 + 2t + 1)}_{M}dt + \underbrace{(4x^3 + 4tx)}_{N}dx = 0$$

sheet #9

Exactness condition is $\boxed{M_x = N_t}$

$$\frac{\partial M}{\partial x} = \frac{\partial}{\partial x}(2x^2+2t+1) = 4x + 0 + 0 = 4x$$
$$\frac{\partial N}{\partial t} = \frac{\partial}{\partial t}(4x^3+4tx) = 0 + 4x = 4x$$

Agree! $\Downarrow$ exact!

Now find a function $f(x,y)$ such that $f_x = N$ & $f_y = M$...

$$f = \int N\,dx = \int (4x^3 + 4tx)\,dx = \frac{4x^4}{4} + \frac{4tx^2}{2} + g(t)$$
$$= x^4 + 2tx^2 + g(t)$$

compare those!

$$f = \int M\,dt = \int (2x^2+2t+1)\,dt = 2tx^2 + t^2 + t + h(x)$$

$$\Rightarrow \boxed{f(x,y) = 2tx^2 + x^4 + t^2 + t}$$ ...and so the solutions to the ODE are level curves $\boxed{f = c}$ i.e.

c a const.

SOLUTIONS $\boxed{2tx^2 + x^4 + t^2 + t = c}$

Solve the following non-constant coefficient linear ODE.

$$(x^2 + 4)\frac{dy}{dx} + 3xy = x$$

$$(*) \quad \boxed{\frac{dy}{dx} + \frac{3x}{x^2+4}y = \frac{x}{x^2+4}}$$

$$\Rightarrow P = \frac{3x}{x^2+4} \Rightarrow \int P\,dx = \int \frac{3x}{x^2+4}\,dx$$
$$= \frac{3}{2}\ln(x^2+4)$$

sheet #8

$$\Rightarrow \text{Integrating factor}, I = e^{\int P\,dx} = e^{\frac{3}{2}\ln(x^2+4)} = e^{\ln((x^2+4)^{3/2})} = (x^2+4)^{3/2}$$

$$(x^2+4)^{3/2}\cdot(*) \Rightarrow (x^2+4)^{3/2}\frac{dy}{dx} + 3x\sqrt{x^2+4}\,y = x\sqrt{x^2+4}$$

$$\Rightarrow \frac{d}{dx}\left((x^2+4)^{3/2}y\right) = x\sqrt{x^2+4}$$

$$\Rightarrow (x^2+4)^{3/2}y = \int x\sqrt{x^2+4}\,dx = \frac{1}{3}(x^2+4)^{3/2} + C$$

$$\Rightarrow y = \frac{1}{(x^2+4)^{3/2}}\left(\frac{1}{3}(x^2+4)^{3/2} + C\right) \Rightarrow \boxed{y = \frac{1}{3} + \frac{C}{(x^2+4)^{3/2}}}$$

Q2]...[**10 points**] Use the characteristic equation method to find the general solution to the following homogeneous ODE. Note that $x = x(t)$. Show all your work.

$$(D^2 - 1)x = 0$$

Sheet #16

Let $x = e^{rt} \Rightarrow Dx = re^{rt} \Rightarrow D^2 x = r^2 e^{rt}$

$\Rightarrow (D^2-1)x = 0$ becomes $(r^2-1)e^{rt} = 0 \Rightarrow r^2 - 1 = 0$ char. Eq^n

$\Rightarrow r^2 = 1 \Rightarrow r = \pm 1$

$\Rightarrow$ Two independent solutions are $e^{1t} = e^t$ and $e^{-1.t} = e^{-t}$

$\Rightarrow$ General sol^n is $\boxed{x = c_1 e^t + c_2 e^{-t}}$

Use the method of undetermined coefficients to find a particular solution to the non-homogeneous ODE. Show all your work.

$$(D^2 - 1)x = t + e^t$$

Sheet #17

$D^2 t = D1 = 0$; $(D-1)e^t = e^t - e^t = 0 \Rightarrow D^2(D-1)(t+e^t) = 0$.

Let x_p be a particular sol^n of $(D^2-1)x_p = t + e^t$

Then $D^2(D-1)(D^2-1)x_p = D^2(D-1)(t+e^t) = 0$

ie. x_p is a solution of the higher order homogeneous equation

$$D^2(D-1)(D^2-1)x_p = 0$$

$r^2(r-1)^2(r+1) = 0 \Rightarrow$ solutions are combination of

$1, t, e^t, te^t, e^{-t}$

Ignore these, because $(D^2-1)(e^{\pm t}) = 0$

$\Rightarrow x_p = a + bt + cte^t$

$\Rightarrow x_p' = b + ce^t + cte^t$

$\Rightarrow x_p'' = 2ce^t + cte^t$

$\Rightarrow x_p'' - x_p = t + e^t$ becomes

$2ce^t - a - bt = 1.e^t + 1.t + 0$

$\Rightarrow \boxed{a=0}$ $\boxed{b=-1}$ & $\boxed{c=\frac{1}{2}}$

$\boxed{x_p(t) = \frac{te^t}{2} - t}$ sol^n

Q3]... [12 points] Write down general solutions to the following linear homogeneous differential equations. In all cases $x = x(t)$ is a function of t.

all 4 are... Sheet #16

$$(D^2 + 6D + 5)x = 0$$

$r^2 + 6r + 5 = 0$ $(r+5)(r+1) = 0$

$r = -5$ $r = -1$

$$\boxed{X(t) = C_1 e^{-5t} + C_2 e^{-t}}$$

$$(D^2 + 4)x = 0$$

$r^2 + 4 = 0$ $\Rightarrow r^2 = -4$ $\Rightarrow r = \pm 2i$

$e^{2it} = \underline{\cos(2t)} + i\ \underline{\sin(2t)}$

$$\Rightarrow \boxed{X(t) = C_1 \cos(2t) + C_2 \sin(2t)}$$

$$(D^2 - 4)^2 x = 0$$

$(r^2 - 4)^2 = 0$ $\Rightarrow r^2 = 4$

$\Rightarrow r = \pm 2$ each w/ multiplicity 2.

$$\boxed{X(t) = C_1 e^{2t} + C_2 t e^{2t} + C_3 e^{-2t} + C_4 t e^{-2t}}$$

$$(D^2 + 2D + 5)x = 0$$

$r^2 + 2r + 5 = 0$

$r = \dfrac{-2 \pm \sqrt{(2)^2 - 4(5)}}{2}$

$= \dfrac{-2 \pm \sqrt{-16}}{2}$

$= \dfrac{-2 \pm 4i}{2}$

$= -1 \pm 2i$

$e^{(-1+2i)t} = e^{-t} e^{2it}$

$= \underline{e^{-t} \cos(2t)} + i\ \underline{e^{-t} \sin(2t)}$

$$\boxed{X(t) = C_1 e^{-t} \cos(2t) + C_2 e^{-t} \sin(2t)}$$

Q4]... **[12 points]** Find the eigenvalues of the following matrix

$$A = \begin{pmatrix} -3 & -2 \\ 1 & 0 \end{pmatrix}$$

$\det(A - \lambda I) = 0$

$\Rightarrow \det \begin{pmatrix} -3-\lambda & -2 \\ 1 & 0-\lambda \end{pmatrix} = 0$

$\Rightarrow (-3-\lambda)(-\lambda) - (1)(-2) = 0$

$\Rightarrow \lambda^2 + 3\lambda + 2 = 0$

$\Rightarrow (\lambda+2)(\lambda+1) = 0$

$\Rightarrow \boxed{\lambda = -2}$ & $\boxed{\lambda = -1}$

e-values

Sheet # 22

Find eigenvectors of A corresponding to each of the eigenvalues above.

$\boxed{\lambda = -2}$

$(A - \lambda I)\vec{v} = \vec{0}$

$\begin{pmatrix} -1 & -2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow x + 2y = 0$

$x = -2y$ $\quad \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$\boxed{\lambda = -1}$

$(A - \lambda I)\vec{v} = \vec{0}$

$\begin{pmatrix} -2 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$x + y = 0$

$y = -x$ $\quad \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Use the eigenvalue–eigenvector method to find the general solution to the homogeneous linear system

$$\frac{d\mathrm{x}}{dt} = A\mathrm{x}$$

where A is the matrix in the first part of this question.

$$\boxed{\vec{X}(t) = c_1 e^{-2t}\begin{pmatrix} -2 \\ 1 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

Q5]... **[10 points]** Consider a linear system of the form

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}$$

where $\mathbf{x}$ is a 2×1 column vector, A is a 2×2 matrix. Suppose that A has one eigenvalue λ with multiplicity 2, and that $\mathbf{u}_1$ is a λ–eigenvector and $\mathbf{u}_2$ is a generalized λ–eigenvector satisfying $(A-\lambda I)\mathbf{u}_2 = \mathbf{u}_1$.

$$\Rightarrow \boxed{A\vec{u}_2 = \lambda\vec{u}_2 + \vec{u}_1} \quad (**)$$

Write down the general solution to the linear system above (your answer will involve the unknown vectors $\mathbf{u}_1$ and $\mathbf{u}_2$). Check that your solution satisfies the equation.

$$\boxed{\vec{X}(t) = C_1 e^{\lambda t}\vec{u}_1 + C_2 e^{\lambda t}(\vec{u}_2 + t\,\vec{u}_1)}$$

Sheet #22 before Euler's identity

Check

$$\frac{d}{dt}(e^{\lambda t}\vec{u}_1) = \lambda e^{\lambda t}\vec{u}_1 = e^{\lambda t}(\lambda\vec{u}_1) = e^{\lambda t} A\vec{u}_1 = A(e^{\lambda t}\vec{u}_1)$$

and

$$\frac{d}{dt} e^{\lambda t}(\vec{u}_2 + t\vec{u}_1) = \left(\frac{de^{\lambda t}}{dt}\right)(\vec{u}_2 + t\vec{u}_1) + e^{\lambda t}\frac{d}{dt}(\vec{u}_2 + t\vec{u}_1) = \lambda e^{\lambda t}(\vec{u}_2 + t\vec{u}_1) + e^{\lambda t}\vec{u}_1$$

$$A e^{\lambda t}(\vec{u}_2 + t\vec{u}_1) = e^{\lambda t} A(\vec{u}_2 + t\vec{u}_1) = e^{\lambda t} A\vec{u}_2 + e^{\lambda t}\, t A\vec{u}_1$$

$$= e^{\lambda t}(\lambda\vec{u}_2 + \vec{u}_1) + \lambda e^{\lambda t}\, t\,\vec{u}_1 \quad \text{...by } (**)$$

agree!!

Consider a linear system of the form

$$\frac{d\mathbf{x}}{dt} = B\mathbf{x}$$

where $\mathbf{x}$ is a 3×1 column vector, B is a 3×3 matrix. Suppose that B has one eigenvalue λ with multiplicity 3, and that $\mathbf{v}_1$ is a λ–eigenvector and $\mathbf{v}_2$ and $\mathbf{v}_3$ are generalized λ–eigenvectors satisfying $(B-\lambda I)\mathbf{v}_2 = \mathbf{v}_1$ and $(B-\lambda I)\mathbf{v}_3 = \mathbf{v}_2$.

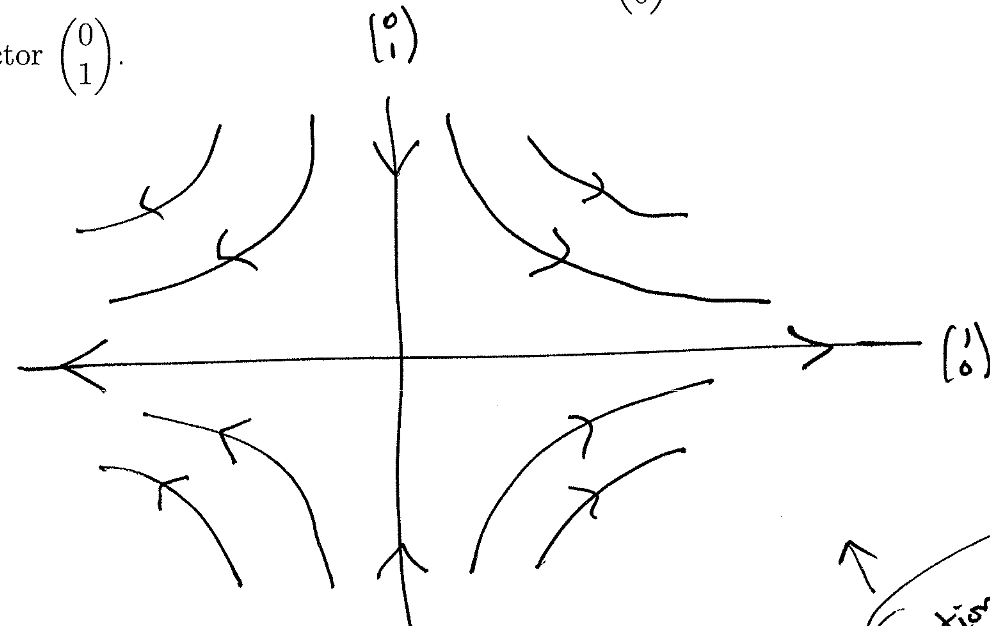
Write down the general solution to the linear system above (your answer will involve the unknown vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$).

$$\vec{X}(t) = C_1 e^{\lambda t}\vec{V}_1 + C_2 e^{\lambda t}(\vec{V}_2 + t\vec{V}_1) + C_3 e^{\lambda t}\left(\vec{V}_3 + t\vec{V}_2 + \frac{t^2}{2}\vec{V}_1\right)$$

& the verification (check) that it's a solution is very similar to the above (but not asked here!)

Q6]... [10 points] Draw a sketch of the solution curves (phase plane portrait) of the linear system $\mathbf{x}' = A\mathbf{x}$ where A is a 2×2 matrix with the following eigenvalues/eigenvectors. Give some reasons to justify your diagrams.

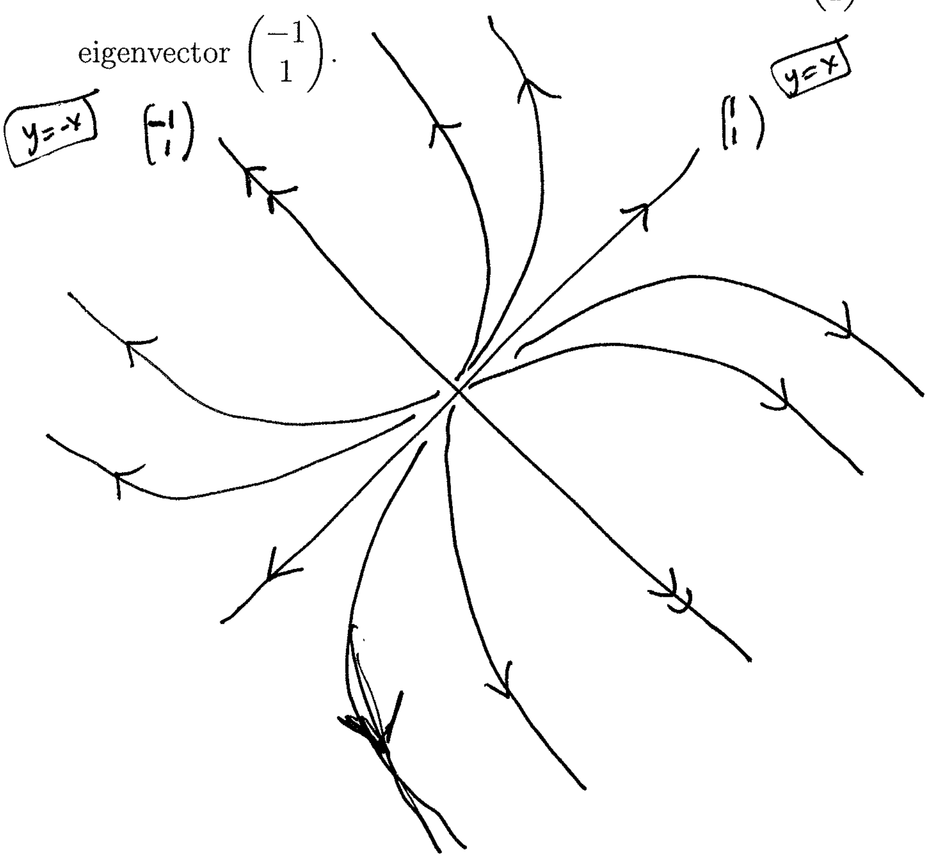
1. A has one eigenvalue equal to 2 with eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and a second eigenvalue equal to -2 and eigenvector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.



Variation on Mid III problem

Just switching eigenvectors between the 2 parts!

2. A has one eigenvalue equal to 2 with eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and a second eigenvalue equal to 5 and eigenvector $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.



Q7]... **[12 points]** Use the method of variation of parameters to find a particular solution to the system

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} e^t \\ e^t \end{pmatrix}$$

and check that your answer is indeed a solution. If it helps, you may assume that the fundamental matrix of the associated homogeneous system is

$$\Phi(t) = \begin{pmatrix} 2e^t & e^{2t} \\ e^t & e^{2t} \end{pmatrix}$$

<u>From $\boxed{\text{sheet \#24}}$</u> ... we have $x_p(t) = \Phi(t) \int \Phi(t)^{-1} f(t)\, dt$

$$\Phi(t)^{-1} f(t) = \begin{pmatrix} 2e^t & e^{2t} \\ e^t & e^{2t} \end{pmatrix}^{-1}\begin{pmatrix} e^t \\ e^t \end{pmatrix} = \frac{1}{2e^{3t} - e^{3t}}\begin{pmatrix} e^{2t} & -e^{2t} \\ -e^t & 2e^t \end{pmatrix}\begin{pmatrix} e^t \\ e^t \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$= \frac{1}{e^{3t}}\begin{pmatrix} 0 \\ e^{2t} \end{pmatrix} = \begin{pmatrix} 0 \\ e^{-t} \end{pmatrix}.$$

$$\int \Phi(t)^{-1} f(t)\,dt = \int \begin{pmatrix} 0 \\ e^{-t} \end{pmatrix} dt = \begin{pmatrix} \int 0\, dt \\ \int e^{-t} dt \end{pmatrix} = \begin{pmatrix} 1 \\ -e^{-t} \end{pmatrix}$$

Finally, $\vec{X}_p(t) = \Phi(t) \int \Phi(t)^{-1} f(t)\, dt = \begin{pmatrix} 2e^t & e^{2t} \\ e^t & e^{2t} \end{pmatrix}\begin{pmatrix} 1 \\ -e^{-t} \end{pmatrix}$

$$= \begin{pmatrix} e^t \\ 0 \end{pmatrix} \qquad \boxed{\vec{X}_p(t) = \begin{pmatrix} e^t \\ 0 \end{pmatrix}}$$

<u>Check</u> $\frac{d\vec{x}_p}{dt} = \frac{d}{dt}\begin{pmatrix} e^t \\ 0 \end{pmatrix} = \begin{pmatrix} e^t \\ 0 \end{pmatrix} \overset{??}{=} \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} e^t \\ 0 \end{pmatrix} + \begin{pmatrix} e^t \\ e^t \end{pmatrix}$

$$= \begin{pmatrix} 0 \\ -e^t \end{pmatrix} + \begin{pmatrix} e^t \\ e^t \end{pmatrix} = \begin{pmatrix} e^t \\ 0 \end{pmatrix} \quad \checkmark \text{ yes!}$$

Q8]... **[10 points]** Use the method of Laplace transforms to solve the IVP

$$x' - x = 2\sin t \qquad x(0) = 0$$

$$\mathcal{L} \Rightarrow \quad sX(s) - \cancel{x(0)}^{\,0} - X(s) = 2\,\mathcal{L}\{\sin t\} = 2\left(\frac{1}{s^2+1}\right)$$

Sheet # 26 (f) Sheet # 26(a) Sheet # 26(d)

$$\Rightarrow (s-1)X(s) = \frac{2}{s^2+1} \quad \Rightarrow \quad \boxed{X(s) = \frac{2}{(s^2+1)(s-1)}}$$

Partial Fractions

$$\frac{2}{(s^2+1)(s-1)} = \frac{As+B}{s^2+1} + \frac{C}{s-1}$$

Framework = $\boxed{\text{sheet \# 27}}$

$$\Rightarrow 2 = (As+B)(s-1) + C(s^2+1)$$

$$0.s^2 + 0.s + 2 = (A+C)s^2 + (B-A)s + (C-B)$$

$A + C = 0 \Rightarrow A = -C$

$B - A = 0 \Rightarrow A = B$

$C - B = 2 \Rightarrow C + C = 2 \Rightarrow \boxed{C = 1}$ $\quad \boxed{A = B = -1}$

$$\Rightarrow X(s) = \frac{-s-1}{s^2+1} + \frac{1}{s-1} = -\frac{s}{s^2+1} - \frac{1}{s^2+1} + \frac{1}{s+1}$$

Sheet # 26(e) Sheet # 26(d) Sheet # 26 (c)

$$\mathcal{L}^{-1} \Rightarrow \quad \boxed{x(t) = -\cos(t) - \sin(t) + e^{-t}}$$

Q9]... **[12 points]** Use the table of Laplace transforms to write down $\mathcal{L}\{\sin t\}$.

$$\mathcal{L}\{\sin t\} = \frac{1}{s^2+1} \qquad \text{sheet \# 26(d)}$$

Now use the table of Laplace transforms to determine $\mathcal{L}\{t \sin t\}$.

$$\text{sheet \# 26(g)} \Rightarrow \quad \mathcal{L}\{t \sin t\} = (-1)\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = (-1)(-1)(s^2+1)^{-2}\cdot(2s)$$

$$= \frac{2s}{(s^2+1)^2}$$

$$\boxed{\mathcal{L}\{t \sin t\} = \frac{2s}{(s^2+1)^2}} \quad —(**)$$

Use the method of Laplace transforms to solve the IVP

$$(D^2+1)x = \cos t \qquad x(0)=0, \qquad x'(0)=0$$

Framework = sheet #27

sheet#26(f) (twice)

sheet

$$\mathcal{L} \Rightarrow \quad s^2 X(s) - \cancel{s\,x(0)}^{\,0} - \cancel{x'(0)}^{\,0} + X(s) = \mathcal{L}\{\cos t\} = \frac{s}{s^2+1}$$

$$\Rightarrow \quad (s^2+1)\,X(s) = \frac{s}{s^2+1}$$

Recognize this from (**)

$$\Rightarrow \quad X(s) = \frac{s}{(s^2+1)^2} = \frac{1}{2}\left(\frac{2s}{(s^2+1)^2}\right)$$

$$\mathcal{L}^{-1} \Rightarrow \quad \boxed{x(t) = \frac{1}{2}t\sin(t)}$$

Note this is Resonance phenomenon — & the ODE represents vibration — free (undamped) with periodic driving force whose frequency = frequency of system.

Note $\quad X(s) = \left(\frac{s}{s^2+1}\right)\left(\frac{1}{s^2+1}\right) \Rightarrow x(t) = \cos(t) * \sin(t)$

$= \left(\text{after painful computation of } \int_0^t \cos(u)\sin(t-u)\,du\right) \dots \frac{t\sin t}{2}$