Monday 05/09/2016 Final Examination 120 minutes

Name: | | Student ID: | |

Instructions.

1. Attempt all questions.
2. Do not write on back of exam sheets. Extra paper is available if you need it.
3. **Show all the steps of your work clearly. A final answer on its own is not enough for full points.**

Question	Points	Your Score
Q1	12	
Q2	10	
Q3	12	
Q4	12	
Q5	10	
Q6	10	
Q7	12	
Q8	10	
Q9	12	
TOTAL	100	

Q1]... [12 points] Test to see if the following ODE is exact. If it is exact, then find a solution.

$$(2x^2 + 2t + 1)dt + (4x^3 + 4tx)dx \;=\; 0$$

Solve the following non-constant coefficient linear ODE.

$$(x^2 + 4)\frac{dy}{dx} + 3xy \;=\; x$$

Q2]... [10 points] Use the characteristic equation method to find the general solution to the following homogeneous ODE. Note that $x = x(t)$. Show all your work.

$$(D^2 - 1)x \;=\; 0$$

Use the method of undetermined coefficients to find a particular solution to the non-homogeneous ODE. Show all your work.

$$(D^2 - 1)x \;=\; t + e^t$$

Q3]... [12 points] Write down general solutions to the following linear homogeneous differential equations. In all cases $x = x(t)$ is a function of t.

$$(D^2 + 6D + 5)x = 0$$

$$(D^2 + 4)x = 0$$

$$(D^2 - 4)^2 x = 0$$

$$(D^2 + 2D + 5)x = 0$$

Q4]. . . [12 points] Find the eigenvalues of the following matrix

$$A = \begin{pmatrix} -3 & -2 \\ 1 & 0 \end{pmatrix}$$

Find eigenvectors of A corresponding to each of the eigenvalues above.

Use the eigenvalue–eigenvector method to find the general solution to the homogeneous linear system

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}$$

where A is the matrix in the first part of this question.

Q5]. . . [10 points] Consider a linear system of the form

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}$$

where $\mathbf{x}$ is a 2×1 column vector, A is a 2×2 matrix. Suppose that A has one eigenvalue λ with multiplicity 2, and that $\mathbf{u}_1$ is a λ–eigenvector and $\mathbf{u}_2$ is a generalized λ–eigenvector satisfying $(A - \lambda I)\mathbf{u}_2 = \mathbf{u}_1$.

Write down the general solution to the linear system above (your answer will involve the unknown vectors $\mathbf{u}_1$ and $\mathbf{u}_2$). Check that your solution satisfies the equation.

Consider a linear system of the form

$$\frac{d\mathbf{x}}{dt} = B\mathbf{x}$$

where $\mathbf{x}$ is a 3×1 column vector, B is a 3×3 matrix. Suppose that B has one eigenvalue λ with multiplicity 3, and that $\mathbf{v}_1$ is a λ–eigenvector and $\mathbf{v}_2$ and $\mathbf{v}_3$ are generalized λ–eigenvectors satisfying $(B - \lambda I)\mathbf{v}_2 = \mathbf{v}_1$ and $(B - \lambda I)\mathbf{v}_3 = \mathbf{v}_2$.

Write down the general solution to the linear system above (your answer will involve the unknown vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$).

Q6]... [10 points] Draw a sketch of the solution curves (phase plane portrait) of the linear system $\mathbf{x}' = A\mathbf{x}$ where A is a 2×2 matrix with the following eigenvalues/eigenvectors. Give some reasons to justify your diagrams.

1. A has one eigenvalue equal to 2 with eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and a second eigenvalue equal to -2 and eigenvector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

2. A has one eigenvalue equal to 2 with eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and a second eigenvalue equal to 5 and eigenvector $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

Q7]...[12 points] Use the method of variation of parameters to find a particular solution to the system

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} e^t \\ e^t \end{pmatrix}$$

and check that your answer is indeed a solution. If it helps, you may assume that the fundamental matrix of the associated homogeneous system is

$$\Phi(t) = \begin{pmatrix} 2e^t & e^{2t} \\ e^t & e^{2t} \end{pmatrix}$$

Q8]. . . [10 points] Use the method of Laplace transforms to solve the IVP

$$x' - x = 2\sin t \qquad\qquad x(0) = 0$$

Q9]. . . [12 points] Use the table of Laplace transforms to write down $\mathcal{L}\{\sin t\}$.

Now use the table of Laplace transforms to determine $\mathcal{L}\{t \sin t\}$.

Use the method of Laplace transforms to solve the IVP

$$(D^2 + 1)x \; = \; \cos t \qquad\qquad x(0) = 0, \qquad x'(0) = 0$$