Exact Equations – Method

A first order ODE of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} \;=\; 0 \qquad (A)$$

or equivalently (using the notation of differentials)

$$M(x,y)dx + N(x,y)dy \;=\; 0 \qquad (B)$$

is said to be *exact* if M and N satisfy the following test:

$$\frac{\partial M}{\partial y} \;=\; \frac{\partial N}{\partial x} \qquad (C)$$

In this case (under mild continuity and differentiability assumptions) we can find a function $F(x,y)$ such that $\frac{\partial F}{\partial x} = M$ and $\frac{\partial F}{\partial y} = N$. Do this by anti-differentiating and comparing:

$$F(x,y) \;=\; \int M(x,y)\,dx$$

$$F(x,y) \;=\; \int N(x,y)\,dy$$

Once we have $F(x,y)$ the general solution to the ODE (A) or (B) is given by the one-parameter family of curves

$$F(x,y) \;=\; C.$$

Exact Equations – Rationale

The rationale is obtained by reversing how we think about things. Start with a one-parameter family of curves, and find an ODE that they satisfy. Then compare this ODE with (A) above.

Recall that the level curves (contour lines) of a function $F(x,y)$ of two variables are defined by

$$F(x,y) \;=\; C$$

where C is a constant. By varying C we obtain a one-parameter family of curves in the xy–plane.

Now suppose that the C–level curve of $F(x,y)$ is described parametrically by $(x(t), y(t))$. Then $F(x(t), y(t)) = C$ and differentiating across by t gives

$$\frac{dF(x(t),y(t))}{dt} \;=\; \frac{dC}{dt} \;=\; 0.$$

Using the chain rule on the left side gives

$$\frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} \;=\; 0.$$

Dividing across by $\frac{dx}{dt}$ and recalling that $\frac{dy}{dx} = \frac{dy}{dt}/\frac{dx}{dt}$ for parametric curves, we obtain the following differential equation that the level curves $F(x,y) = C$ satisfy:

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} \;=\; 0.$$

This looks just like (A) above when $M = \frac{\partial F}{\partial x}$ and $N = \frac{\partial F}{\partial y}$. Now it is easy to see where the test (C) comes from; it is a manifestation of Clairaut's theorem on mixed partial derivatives:

$$\frac{\partial M}{\partial y} \;=\; \frac{\partial^2 F}{\partial y \partial x} \;=\; \frac{\partial^2 F}{\partial x \partial y} \;=\; \frac{\partial N}{\partial x}.$$