Complete the truth table for the following two expressions.

P	Q	R	$(P \lor Q) \to R$	$(P \to R) \land (Q \to R)$
T	T	T	T	T
T	T	F	F	F
T	F	T	T	T
T	F	F	F	F
F	T	T	T	T
F	T	F	F	F
F	F	T	T	T
F	F	F	T	T

Definition. An integer a is said to be *divisible by 3* if $a = 3q$ for some $q \in \mathbb{Z}$.

Remark. As was the case even and odd integers, we can highlight every third integer, and then get a simple description of the integers that are not divisible by 3.

Definition. An integer a is said to be *not divisible by 3* if there exists an integer $p \in \mathbb{Z}$ and either $a = 3p + 1$ or $a = 3p + 2$.

Prove the following statements.

Theorem. If an integer a is divisible by 3, then a^2 is divisible by 3.

Proof. By hypothesis a is divisible by 3. This means that $a = 3p$ for some $p \in \mathbb{Z}$. Therefore,

$$a^2 = (3p)^2 = 9p^2 = 3(3p^2)$$

Now, by closure of $\mathbb{Z}$ under multiplication $q = 3p^2 \in \mathbb{Z}$. Thus $a^2 = 3q$ is divisible by 3. $\square$

Theorem. If an integer a is not divisible by 3, then a^2 is not divisible by 3.

Proof. By hypothesis a is not divisible by 3. This means that $a = 3p + 1$ for some $p \in \mathbb{Z}$ or $a = 3p + 2$ for some $p \in \mathbb{Z}$. There are two cases to consider.
Case 1. $a = 3p + 1$ for some $p \in \mathbb{Z}$.
In this case

$$a^2 = (3p + 1)^2 = 9p^2 + 6p + 1 = 3(3p^2 + 2p) + 1$$

Now, by closure of $\mathbb{Z}$ under multiplication and addition $q = 3p^2 + 2p \in \mathbb{Z}$. Thus $a^2 = 3q + 1$ is not divisible by 3.
Case 2. $a = 3p + 2$ for some $p \in \mathbb{Z}$.
In this case

$$a^2 = (3p + 2)^2 = 9p^2 + 12p + 4 = 3(3p^2 + 4p + 1) + 1$$

Now, by closure of $\mathbb{Z}$ under multiplication and addition $q = 3p^2 + 4p + 1 \in \mathbb{Z}$. Thus $a^2 = 3q + 1$ is not divisible by 3.
In each case, we conclude that a^2 is not divisible by 3. $\square$

Note the logic of your argument. Let P be the statement "a is not divisible by 3," and let R be the statement "a^2 is not divisible by 3."

- You want to prove $P \to R$.
- You know from the second definition that $P \to (Q_1 \vee Q_2)$ where Q_1 is the statement "$a = 3p+1$ for some $p \in \mathbb{Z}$" and Q_2 is the statement "$a = 3p + 2$ for some $p \in \mathbb{Z}$."
- You know from the tautology $[(P \to Q) \wedge (Q \to R)] \to (P \to R)$ that $(P \to R)$ will follow once you show that $(Q_1 \vee Q_2) \to R$. [Here Q is replaced by the compound statement $Q_1 \vee Q_2$]
- You know from the logical equivalence proven earlier that $(Q_1 \vee Q_2) \to R$ is equivalent to $(Q_1 \to R) \wedge (Q_2 \to R)$. This means that you have to prove the two cases $(Q_1 \to R)$ and $(Q_2 \to R)$ separately.
- This is the logical framework behind a "proof by cases." Normally, one just states that there are two (or more) cases to consider, and then proceeds to consider them separately in a proof. It is comforting to know that all this is consistent with the laws of logic.

Take stock of where we are.

- We have some theorems about integers under our belts.

  (I) If a is an even integer, then a^2 is even.

  (II) If a is an odd integer, then a^2 is odd.

  (III) If a is an integer which is divisible by 3, then a^2 is divisible by 3.

  (IV) If a is an integer which is not divisible by 3, then a^2 is not divisible by 3.

- We have some logical equivalences of implication.

  1. The conditional statement $(P \to Q)$ is equivalent to its *contrapositive* $(\neg Q) \to (\neg P)$.
  2. The conditional statement $(P \to Q)$ is equivalent to $(\neg P \lor Q)$.

Prove the following new theorems about integers.

Theorem 1. Let a be an integer. If a^2 is even, then a is even.

Proof. This is the contrapositive of (II) above. □

Theorem 2. Let a be an integer. If a^2 is odd, then a is odd.

Proof. This is the contrapositive of (I) above. □

Theorem 3. Let a be an integer. a^2 is even iff (if and only if) a is even.

Proof. This follows from Theorem 1 and (I). □

Theorem 4. Let a be an integer. a^2 is odd iff a is odd.

Proof. This follows from Theorem 2 and (II). □

Theorem 5. Let a be an integer. If a^2 is divisible by 3, then a is divisible by 3.

Proof. This is the contrapositive of (IV) above. □

Theorem 6. Let a be an integer. If a^2 is not divisible by 3, then a is not divisible by 3.

Proof. This is the contrapositive of (III) above. □

Theorem 7. Let a be an integer. a^2 is divisible by 3 iff a is divisible by 3.

Proof. This follows from Theorem 5 and (III). □

Theorem 8. Let a be an integer. a^2 is not divisible by 3 iff a is not divisible by 3.

Proof. This follows from Theorem 6 and (IV). □

Think about these results. What is special about 2 and 3. Are there similar theorems for 4, 5, 6,....?

More on this anon...

Complete the truth table for the following two expressions.

P	Q	R	$P \rightarrow (Q \vee R)$	$(P \wedge \neg Q) \rightarrow R$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	T
F	T	F	T	T
F	F	T	T	T
F	F	F	T	T

Prove the following theorem about real numbers. Just use the properties of real numbers listed in the textbook in Table 1.2 on page 18.

Theorem. If x and y are non-zero real numbers, then xy is non-zero.

- Write down the contrapositive statement.
- Note that the contrapositive is in the form $P \to (Q \vee R)$ for suitable statements P, Q, R.
- Write down the logically equivalent statement $(P \wedge \neg Q) \to R$.
- Prove this statement working from its hypotheses and using the properties in Table 1.2 of the textbook.

This is covered in detail in a separate handout. See that one for details.

Remark. Why the big deal about this result? Isn't the product of two non-zero quantities always non-zero?
Ehhh...not so fast. Answering these questions carefully will lead you into the wonderful world of *ring theory* where you will meet strange animals called *zero divisors.* Our journey will be taken very gently; via *modular arithmetic.*