Cube

$$C = [-1,1]^3 = [-1,1]\times[-1,1]\times[-1,1]$$
$$= \{(x,y,z)\in\mathbb{R}^3 \mid -1\le x\le 1,\ -1\le y\le 1,\ -1\le z\le 1\} \subseteq \mathbb{R}^3.$$

Face

$$F_1 = \{1\}\times[-1,1]^2$$
$$= \{(x,y,z)\in\mathbb{R}^3 \mid x=1,\ -1\le y\le 1,\ -1\le z\le 1\} \subseteq C$$

is a face of the cube C.

Isometry

A function
$$f: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$
$$: (x,y,z) \longmapsto f(x,y,z) = \big(f_1(x,y,z),\ f_2(x,y,z),\ f_3(x,y,z)\big)$$

is an <u>Isometry</u> if it preserves distance; that is

$$d\big(f(x,y,z),\ f(a,b,c)\big) = d\big((x,y,z),(a,b,c)\big)$$ for all points $(x,y,z)\in\mathbb{R}^3$ & $(a,b,c)\in\mathbb{R}^3$.

Where $d\big((x,y,z),(a,b,c)\big) = \sqrt{(x-a)^2+(y-b)^2+(z-c)^2}$.

Note $d\big(f(x,y,z),\ f(a,b,c)\big) = \Big[\big(f_1(x,y,z)-f_1(a,b,c)\big)^2 + \big(f_2(x,y,z)-f_2(a,b,c)\big)^2 + \big(f_3(x,y,z)-f_3(a,b,c)\big)^2\Big]^{\frac{1}{2}}$

Symm(C)

The group $G = \mathrm{Symm}(C)$ of symmetries of the cube C is

$$G = \{f \mid f:\mathbb{R}^3\to\mathbb{R}^3 \text{ is an isometry, and } f(C) = C\}$$

For example $\quad f: \mathbb{R}^3 \to \mathbb{R}^3 : (x,y,z) \longmapsto (x, y, -z)$

is Reflection in the plane $z=0$.

$f_1(x,y,z) = x$
$f_2(x,y,z) = y$
$f_3(x,y,z) = -z$

$f_i : \mathbb{R}^3 \to \mathbb{R}$

Coordinate functions

Verify it's an isometry

$$d(f(x,y,z),\, f(a,b,c)) = d((x,y,-z),(a,b,-c))$$

$$= \sqrt{(x-a)^2 + (y-b)^2 + (-z-(-c))^2}$$

factor out $(-1)^2 = 1$

$$= \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}$$

$$= d((a,b,c),(x,y,z)) \; \boxed{\checkmark}$$

If $(x,y,z) \in C$ then
$-1 \le x \le 1$
$-1 \le y \le 1$
$-1 \le z \le 1 \Rightarrow -1 \le -z \le -(-1)$
$-1 \le z \le 1$

times(-1)

ie $f(x,y,z) \in C$ too.

$\Rightarrow f(C) = C$.

$\Rightarrow f \in \mathrm{Symm}(C)$.

Let $\boxed{H = \{ f \in \mathrm{Symm}(C) \mid f(F_1) = F_1 \}}$ Def^n of H

eg Reflection in $z=0$ above is in H $\longrightarrow$

$$(x, y, z) \in F_1 \Rightarrow x = 1, \ -1 \le y \le 1, \ -1 \le z \le 1$$

$$\Rightarrow x = 1, \ -1 \le y \le 1, \ -1 \le -z \le -(-1)$$

$$\Rightarrow (1, y, -z) \in F_1$$

$$\Rightarrow f(1, y, z) \in F_1$$

$$\Rightarrow f(F_1) = F_1$$

$$\Rightarrow f \in H \ \square$$

<u>Claim</u> $H < \mathrm{Symm}(C)$ is a <u>subgroup</u>.

<u>Pf</u>

- $\mathbb{I} : \mathbb{R}^3 \to \mathbb{R}^3 : (x, y, z) \longmapsto (x, y, z)$ satisfies $\mathbb{I}(F_1) = F_1$

$\Rightarrow \mathbb{I} \in H$ identity.

- If $f, g \in H$ then $g \circ f(F_1) = g(f(F_1))$

$= g(F_1)$ .... $f \in H$

$= F_1$ .... $g \in H$

$\Rightarrow g \circ f \in H$ closure

- $g^{-1}(g(F_1)) = g^{-1}(F_1) \Rightarrow g^{-1}g(F_1) = g^{-1}(F_1)$ inverses

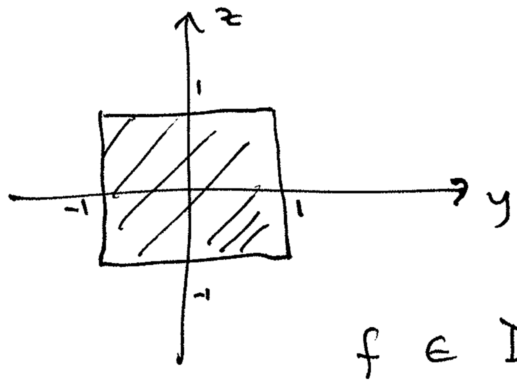
$\Rightarrow \mathbb{I}(F_1) = g^{-1}(F_1)$

$\Rightarrow F_1 = g^{-1}(F_1) \Rightarrow g^{-1} \in H$

Thus, H is a subgroup of $\mathrm{Symm}(C)$. ☒

$H \cong D_4$, and so has $|D_4| = 8$ elements.

<u>Proof</u> Write elements of D_4 out explicitly using yz- coordinates for $\mathbb{R}^2$



$$S = [-1,1] \times [-1,1] \subseteq \mathbb{R}^2$$
$$= \{(y,z) \in \mathbb{R}^2 \mid -1 \le y \le 1,\ -1 \le z \le 1\}$$
is the square.

$$f \in D_4 = \mathrm{Symm}(S)$$

then $f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2 : (y,z) \longmapsto f(y,z) = (f_1(y,z), f_2(y,z))$

is a function which (1) preserves distance
$$d((y,z),(b,c)) = \sqrt{(y-b)^2 + (z-c)^2}$$

and (2) $f(S) = S$.

Then $f \in D_4$ gives rise to an isometry

$$(*) \longrightarrow \left[\begin{array}{l} \hat{f} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \\ \quad : (x, y, z) \longmapsto (x, f(y,z)) = (x, f_1(y,z), f_2(y,z)) \end{array} \right.$$

Such that $\hat{f}(C) = C$.

That is $\hat{f} \in \mathrm{Symm}(C)$.

Note that

$$\widehat{f \circ g} = \hat{f} \circ \hat{g}$$

& $\hat{f}(F_1) = F_1$ since $\hat{f}$ preserves x-coordinate & F_1 is defined by $\underline{\underline{x = 1}}$.

Thus, $f \longmapsto \hat{f}$

defines an isomorphism of groups

$$D_4 \xrightarrow[f \mapsto \hat{f}]{} H < \mathrm{Symm}(C).$$

$$L_{g_1}(H) = L_{g_2}(H) \iff g_1(F_1) = g_2(F_1)$$

OR

$$\boxed{g_1 H = g_2 H \iff g_1(F_1) = g_2(F_1)}$$

Remark: This looks like "almost a tautology", but remember what each side means.

- $g_1 H$, $g_2 H$ are left cosets of the 8-element subgroup H of the finite group $G = \mathrm{Symm}(C)$.
- $g_1(F_1)$, $g_2(F_1)$ are images of the face F_1 under the symmetries g_1, g_2.

In particular, F_1 has ∞ly many points and each $g_i(F_1)$ has ∞ly many points & is equal to one of 6 possibilities:

$\{1\} \times [-1,1] \times [-1,1]$; $[-1,1] \times \{1\} \times [-1,1]$; $[-1,1] \times [-1,1] \times \{1\}$

$\{1\} \times [-1,1] \times [-1,1]$; $[-1,1] \times \{-1\} \times [-1,1]$; $[-1,1] \times [-1,1] \times \{-1\}$

(**)

Proof (of $g_1 H = g_2 H \iff g_1(F_1) = g_2(F_1)$).

"$\Rightarrow$"

$$g_1 H = g_2 H \Rightarrow g_2^{-1}(g_1 H) = g_2^{-1}(g_2 H)$$

$$\Rightarrow (g_2^{-1} g_1) H = (g_2^{-1} g_2) H = \mathbb{I} H = H.$$

Now $g_2^{-1} g_1 = g_2^{-1} g_1 \mathbb{I} \in g_2^{-1} g_1 H$

$= H$ --- told this

$\Rightarrow g_2^{-1} g_1 \in H$

$\Rightarrow g_2^{-1} g_1 (F_1) = F_1$ --- by defⁿ of H.

$\Rightarrow g_2 \left(g_2^{-1} g_1 (F_1) \right) = g_2(F_1)$

$\Rightarrow \underbrace{\cancel{(g_2 g_2^{-1})}}_{\mathbb{I}} \left(g_1 (F_1) \right) = g_2(F_1)$

$\Rightarrow g_1(F_1) = g_2(F_1)$.

one direction proven.

"$\Leftarrow$" $g_1(F_1) = g_2(F_1)$

$$\Rightarrow g_2^{-1}(g_1(F_1)) = g_2^{-1}(g_2(F_1))$$

$$\Rightarrow (g_2^{-1}g_1)(F_1) = (g_2^{-1}g_2)(F_1)$$

$$= \mathbb{I}(F_1)$$

$$= F_1$$

$\Rightarrow g_2^{-1}g_1 \in H$ ... by def$^{=}$ of H.

$\Rightarrow g_2^{-1}g_1 H = H$ ... Left mult by $g_2^{-1}g_1$ just permutes elements of H.

$$\Rightarrow g_2(g_2^{-1}g_1 H) = g_2(H)$$

$$\Rightarrow \underbrace{(g_2 g_2^{-1})}_{\mathbb{I}} g_1 H = g_2 H$$

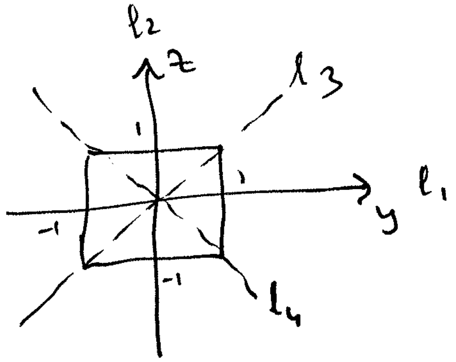
$\Rightarrow g_1 H = g_2 H$ done!

<u>Finally</u> distinct left cosets of H in Symm(C) correspond to distinct images of F_1 in C, & there are 6 of the latter ... list $(**)$ above.

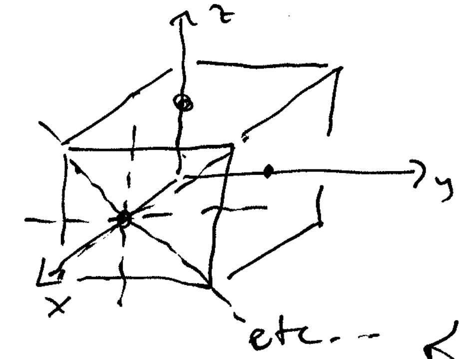
$$\text{Lagrange} \Rightarrow |\text{Symm}(C)| = |H|(\#\text{ left cosets of } H \text{ in Symm}(C))$$

$$= 8(6)$$

$$= 48.$$



$f \longmapsto \hat{f}$



$$l_1 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 : (y,z) \longmapsto (y,-z)$$

$$l_2 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 : (y,z) \longmapsto (-y,z)$$

$$l_3 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 : (y,z) \longmapsto (z,y)$$

$$l_4 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 : (y,z) \longmapsto (-z,-y)$$

$$R : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 : (y,z) \longmapsto (-z,y)$$

$$R^2 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 : (y,z) \longmapsto (-y,-z)$$

$$R^3 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 : (y,z) \longmapsto (z,-y)$$

$$\mathbb{I} = R^4 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 : (y,z) \longmapsto (y,z)$$

The explicit isomorphism

$$D_4 \longrightarrow I < G$$
$$f \longmapsto$$

---

e.g. ...

$$\hat{l}_1 : \mathbb{R}^3 \longrightarrow \mathbb{R}^3 : (x,y,z) \longmapsto (x, l_1(y,z)) = (x,y,-z)$$

$$\hat{l}_3 : \mathbb{R}^3 \longrightarrow \mathbb{R}^3 : (x,y,z) \longmapsto (x, l_3(y,z)) = (x,z,y)$$

$$\hat{R} : \mathbb{R}^3 \to \mathbb{R}^3 : (x,y,z) \longmapsto (x, R(y,z)) = (x,-z,y)$$

etc...