Properties of addition and multiplication of real numbers, and some consequences

Here is a list of properties of addition and multiplication in $\mathbb{R}$. They are given in Table 1.2 on page 18 of your book. It is good practice to "not accept everything that you are told" and to see if you can see why other "obvious sounding" properties of the real numbers follow from the ones given in Table 1.2. This type of critical thinking lead to the development of axiomatic systems; the earliest (and most widely known) being Euclid's axioms for geometry.

The set of real numbers $\mathbb{R}$ is *closed* under addition (denoted by $+$) and multiplication (denoted by $\times$ or simply by juxtaposition) and satisfies:

1. Addition is *commutative.*
$$x + y = y + x$$
for all $x, y \in \mathbb{R}$.

2. Addition is *associative.*
$$(x + y) + z = x + (y + z)$$
for all $x, y, z \in \mathbb{R}$.

3. There is an *additive identity element.* There is a real number 0 with the property that
$$x + 0 = x$$
for all $x \in \mathbb{R}$.

4. Every real number has an *additive inverse.* Given any real number x, there is a real number $(-x)$ so that
$$x + (-x) = 0$$

5. Multiplication is *commutative.*
$$xy = yx$$
for all $x, y \in \mathbb{R}$.

6. Multiplication is *associative.*
$$(xy)z = x(yz)$$
for all $x, y, z \in \mathbb{R}$.

7. There is a *multiplicative identity element.* There is a real number 1 with the property that
$$x.1 = x$$
for all $x \in \mathbb{R}$. Furthermore, $1 \neq 0$.

8. Every non-zero real number has a *multiplicative inverse.* Given any real number $x \neq 0$, there is a real number $\frac{1}{x}$ so that
$$x\frac{1}{x} = 1$$

9. Multiplication *distributes* over addition
$$x(y + z) = xy + xz$$
for all $x, y, z \in \mathbb{R}$.

Immediate consequences. It is easy to see that

$$x + 0 = 0 + x = x$$

(just combine 1 and 3),

$$x + (-x) = (-x) + x = 0$$

(just combine 1 and 4),

$$1x = x1 = x$$

(just combine 5 and 7), and that

$$x\frac{1}{x} = \frac{1}{x}x = 1$$

for non-zero x (just combine 5 and 8). In other words, when using properties (3), (4), (7) and (8) we can have the two terms added (or multiplied) in whatever order we like.

Further consequences. It takes a little more work to see that there is only one additive identity (namely 0). Suppose y were an additive identity. This means that $y+x = x+y = x$ for all numbers x. In particular taking $x = 0$ gives $y + 0 = 0$. But, because 0 is an additive identity we know that $0 + x = x + 0 = x$ for all numbers x. In particular, taking $x = y$ gives $y + 0 = y$. Combining this with the conclusion of the previous sentence gives

$$y = y + 0 = 0$$

and so $y = 0$.

Likewise (give the details yourself) there is only one multiplicative identity (namely 1), every number has a unique additive inverse, and every non-zero number has a unique multiplicative inverse.

Two more consequences. It takes a bit more work to see that the statement $z0 = 0$ for all numbers z follows from the list of properties above. Here goes...

Proposition A. $z0 = 0z = 0$ for all real numbers z.

Proof. By property 3 with $x = 0$ we have

$$0 + 0 = 0$$

Multiply both sides of this equation by an arbitrary real number z to obtain

$$z(0 + 0) = z0$$

Using distributivity (property 9) this becomes

$$z0 + z0 = z0$$

Now, property 4 guarantees that there is a number $-z0$ with the property that

$$z0 + (-z0) = 0$$

Adding this to both sides of the previous equality gives

$$(z0 + z0) + (-z0) = z0 + (-z0)$$

Using associativity (property 2) gives

$$z0 + (z0 + (-z0)) = z0 + (-z0)$$

Now using the fact that $z0 + (-z0) = 0$ we get

$$z0 + 0 = 0$$

Using the fact that 0 is the additive identity, the LHS simplifies to $z0$, and we get

$$z0 = 0$$

Remembering that z is an arbitrary real number, this is what we wanted to prove. □

Finally, here is another obvious sounding statement which follows (using a few more twists of logic) from the results and initial list of properties above.

Propostion B. If $x \neq 0$ and $y \neq 0$ are real numbers, then $xy \neq 0$.

Proof. We will prove the contrapositive statement; namely, if $xy = 0$, then $x = 0$ or $y = 0$. This is logically equivalent (by the equivalence $P \to (Q \vee R) \equiv (P \wedge \neg Q) \to R$) to the following: if $xy = 0$ and $x \neq 0$, then $y = 0$. So we can start from two hypotheses: $xy = 0$ and $x \neq 0$.

Now since $x \neq 0$, by property 8 there is a number $\frac{1}{x}$ such that

$$\frac{1}{x}x = x\frac{1}{x} = 1.$$

Thus

$$\begin{aligned} y &= 1y && \text{by Property (7)} \\ &= (\frac{1}{x}x)y && \text{by the previous sentence} \\ &= \frac{1}{x}(xy) && \text{by Property (6)} \\ &= \frac{1}{x}0 && \text{by the hypothesis } xy = 0 \\ &= 0 && \text{by Proposition A.} \end{aligned}$$

Therefore $y = 0$ and the proposition is proven. □

Remark. Here is an arithmetic system where Proposition B fails to hold, yet where many of the properties 1–9 hold. Which of the properties 1–9 hold for $+_6$ and $\times_6$?

The set of numbers $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ under modular addition ($+_6$) and multiplication ($\times_6$).

$+_6$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

$\times_6$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

The numbers 2, 3, 4 and 6 are called *zero divisors*. Note that they are non-zero themselves, and they do **not** have multiplicative inverses. Note that 1 and 5 have multiplicative inverses, and are not zero divisors. Think about the proof of Proposition B again!