Fa'09: MATH 2513–001 **Discrete Mathematics** **Noel Brady**

Thursday 12/17/2009 Final Exam 8:00am–10:00am

Name: ______________________ Student ID: ______________________

Instructions.

1. Attempt all questions.
2. Do not write on back of exam sheets. Extra paper is available if you need it.
3. Show all the steps of your work clearly.
4. $\mathbb{R}$ is the set of all real numbers.
5. $\mathbb{E}^2$ is the euclidean plane.
6. $\mathbb{Z}$ is the set of all integers, and $\mathbb{Z}^+$ is the set of all positive integers.

Question	Points	Your Score
Q1	10	
Q2	14	
Q3	14	
Q4	14	
Q5	10	
Q6	14	
Q7	24	
TOTAL	100	

Q1]. . . [10 points] State the principle of induction.

Give a proof by induction of the following.

$$1^2 + \cdots + n^2 \;=\; \frac{n(n+1)(2n+1)}{6}, \quad \forall n \in \mathbb{Z}^+.$$

Q2]. . . [14 points] State the Fundamental Theorem of Arithmetic.

Use the Fundamental Theorem to prove that $\log_6(8)$ is irrational.

Use the Fundamental Theorem to prove that the only positive integers whose square roots are rational are the perfect squares: $1, 4, 9, \ldots$

Q3]. . . [20 points] Let $f : A \to B$ be a function, and $S \subset A$ and $T \subset B$ be subsets. Define the *image* $f(S)$ of the subset S and the *preimage* $f^{-1}(T)$ of the subset T.

Prove that $S \subset f^{-1}(f(S))$ for any subset $S \subset A$.

Given an example of a function f and a subset S of the domain, which shows that the above inclusion need not be an equality.

Give a proof that the inclusion is in fact an equality in the case when f is injective.

Q4]. . . [14 points] Write down an expression for $|A \cup B|$, the cardinality of the union of two finite sets A and B, and say why it is true.

Let A, B, and C be finite sets. Write down an expression for the cardinality of the union, $|A \cup B \cup C|$. Give a proof of your result.

How many positive integers less than or equal to 1,000 are divisible by at least one of 5, 7, or 11? *Hint: Let* $A = \{n \in \mathbb{Z}^+ \mid n \leq 1,000 \text{ and } 5|n\}$*...*

Q5]. . . [10 points] Define what it means for a function $f : A \to B$ to be surjective.

Recall that $\mathbb{Z}^+$ denotes the set of positive integers, and that $\{1, 2, 3\}^{\mathbb{Z}^+}$ denotes the set of all functions from $\mathbb{Z}^+$ to the set $\{1, 2, 3\}$. Give a detailed proof that no function

$$g : \mathbb{Z}^+ \;\to\; \{1, 2, 3\}^{\mathbb{Z}^+}$$

can be surjective. (Hint: the usual Cantor diagonal argument)

Q6]. . . [20 points] Use the Euclidean Algorithm to compute $\gcd(64, 46)$. Show all the steps of your work clearly.

Find integers s, t so that $\gcd(64, 46) = s(64) = t(46)$. Show your work clearly.

Find all **integer** solutions x, y to the equation $46x + 64y = 8$.

Q7]. . . [24 points] True or False. Supplying *short* justifications for your choice in each case may help with partial credit.

1. $\neg\forall x(P(x) \rightarrow Q(x)) \;\equiv\; \exists x(P(x) \wedge \neg Q(x))$

2. $P \rightarrow Q \;\equiv\; Q \rightarrow \neg P$

3. $(12)(13)(14)(15)(16) = (654321)$

4. There exists a bijection between $\mathbb{R} \times \mathbb{R}$ and $\mathbb{R}$.

5. There exists a bijection between $\mathbb{R} \times \mathbb{R}$ and the set of all functions from $\mathbb{R}$ to $\mathbb{R}$.

6. The composition of a 90 degree rotation about some point of $\mathbb{E}^2$ and a translation of $\mathbb{E}^2$ gives another 90 degree rotation of $\mathbb{E}^2$.

7. $\forall x(A(x) \vee B(x)) \;\equiv\; (\forall x A(x)) \vee (\forall x B(x))$

8. There exists a bijection between the set $\mathbb{R}$ and the power set of $\mathbb{Z}$.

9. The product of any 4 consecutive positive integers is divisible by 24.

10. For each $n \in \mathbb{Z}^+$, there exists a sequence of n consecutive positive integers, none of which are prime.

11. If $f : A \rightarrow B$ is a function and $S_1, S_2 \subset A$, then $f(S_1 \cap S_2) = f(S_1) \cap f(S_2)$.

12. If A, B, C, D are sets, then $(A \cup B) \times (C \cup D) = (A \times C) \cup (B \times D)$.

Bonus Question. *Only attempt this after you have attempted all the numbered questions.*
Define what $|A| \leq |B|$ means for two (possibly infinite) sets A and B.

State the Schroder-Bernstein Theorem.

Write down an injective function $\mathbb{Z}^+ \to \mathbb{Z}^+ \times \cdots \times \mathbb{Z}^+$ (n copies), and prove that your map is indeed injective.

Let $2, 3, 5, \ldots, p_n$ be the first n prime numbers. Define a function $g : \mathbb{Z}^+ \times \cdots \times \mathbb{Z}^+ \to \mathbb{Z}^+$ by $g(a_1, \ldots, a_n) = 2^{a_1} 3^{a_2} \cdots p_n^{a_n}$. Prove that g is injective. Give the name of any result that you use in your proof.

What can you conclude about the sets $\mathbb{Z}^+$ and $\mathbb{Z}^+ \times \cdots \times \mathbb{Z}^+$ (n copies)?