Friday 12/12/2014 Final Examination 2 hours

Name: ____________________ Student ID: ____________________

Instructions.

1. Attempt all questions.
2. Do not write on back of exam sheets. Extra paper is available if you need it.
3. Show all the steps of your work clearly.

Question	Points	Your Score
Q1	15	
Q2	15	
Q3	15	
Q4	15	
Q5	15	
Q6	15	
Q7	15	
Q8	20	
TOTAL	125	

Q1]. . . [15 points] State the Fundamental Theorem of Arithmetic.

Prove that if p is prime, then the cube root $\sqrt[3]{p}$ is irrational.

Prove that $\log_2 3$ is irrational.

Q2]. . . [15 points] Give the definition of $a \equiv b \mod m$.

Find the remainder when 27,182,818 is divided by 11. Show details of your work.

Find the remainder when 2^{66} is divided by 17. Show details of your work.

Today is Friday. What day of the week will it be 1 million (10^6) days from today? Show details of your work.

Q3]. . . [15 points] What is the definition of the greatest common divisor, (a, b), of non-zero integers a, b?

State a theorem which relates the value of (a, b) to the values of a and b.

Prove that if a, b, c are integers such that $a \mid bc$ and $(a, b) = 1$, then $a \mid c$.

Prove that if p is prime, a, b are integers and $p \mid ab$, then $p \mid a$ or $p \mid b$.

Q4]...[15 points] Give the definition of an injective function.

Give the definition of a surjective function.

Determine if the following functions are only injective, only surjective, both injective and surjective, or neither. Give arguments to support your answers.

- $f : \mathbb{R} \to \mathbb{R} : x \mapsto e^x$.

- $g : \mathbb{Z}_4 \to \mathbb{Z}_5 - \{0\} : x \mapsto 2^x$.

- $h : \mathbb{R}^2 \to \mathbb{R} : (x, y) \mapsto 2x + y$.

Q5]. . . [15 points] State the principle of mathematical induction.

Let n be an integer which is at least 2. Suppose there are n OU fans, each wearing a crimson or a cream t-shirt, standing in line facing a concession stand. Suppose that the fan at the front of the line is wearing a crimson t-shirt and the fan at the end of the line is wearing a cream t-shirt. Use induction to prove that somewhere in the line there is an OU fan wearing a crimson t-shirt standing directly in front of an OU fan wearing a cream t-shirt.

- State the base case, and say why it is true.

- State the induction step. What are you assuming, and what do you have to prove?

- Give a proof of the induction step.

- What is your conclusion?

Q6]... [15 points] State the Schroeder-Bernstein Theorem.

Write down an injective map $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \times \mathbb{N} \times \mathbb{N}$. Verify that your map f is injective.

Write down an injective map $g : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N} \times \mathbb{N}$. Verify that your map g is injective.

Prove that the sets $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ are equivalent.

Q7]... [15 points] Describe the elements of the group G of symmetries of a regular pentagon. How many elements does G have?

Describe an explicit isomorphism between the group G above and a subgroup of $\text{Perm}(\{1, 2, 3, 4, 5\})$.

Q8]... [20 points] Determine whether each of the following are True or False. Give brief reasons for each of your answers.

1. $\exists x(A(x) \wedge B(x))$ is logically equivalent to $\exists x A(x) \wedge \exists x B(x)$.

2. $P \longrightarrow Q$ is logically equivalent to $Q \longrightarrow P$.

3. $\mathcal{P}(\mathcal{P}(\emptyset))$ has one element, where $\mathcal{P}(A)$ denotes the power set of a set A.

4. The order of the permutation $(123)(4567)$ is 6.

5. $\neg(P \longrightarrow Q)$ is logically equivalent to $P \wedge \neg Q$.

6. An element of A^B is a subset of $B \times A$.

7. $\{\emptyset\} - \emptyset = \{\}$.

8. $|A \cup B \cup C| = |A| + |B| + |C| + |A \cap B \cap A \cap C| - |A \cap B| - |A \cap C| - |B \cap C|$.

9. If p is a prime, then $p \mid (m^p - m)$ for all integers m.

10. The set of all functions from $\mathbb{R}$ to $\mathbb{R}$ has the same cardinality as the set $\mathbb{R} \times \mathbb{R}$.