MIDTERM I – solutions

Axioms of addition and multiplication of real numbers

The set of real numbers $\mathbb{R}$ is *closed* under addition (denoted by $+$) and multiplication (denoted by $\times$ or simply by juxtaposition) and satisfies:

1. Addition is *commutative.*

$$x + y = y + x$$

   for all $x, y \in \mathbb{R}$.

2. Addition is *associative.*

$$(x + y) + z = x + (y + z)$$

   for all $x, y, z \in \mathbb{R}$.

3. There is an *additive identity element.* There is a real number 0 with the property that

$$x + 0 = x$$

   for all $x \in \mathbb{R}$.

4. Every real number has an *additive inverse.* Given any real number x, there is a real number $(-x)$ so that

$$x + (-x) = 0$$

5. Multiplication is *commutative.*

$$xy = yx$$

   for all $x, y \in \mathbb{R}$.

6. Multiplication is *associative.*

$$(xy)z = x(yz)$$

   for all $x, y, z \in \mathbb{R}$.

7. There is a *multiplicative identity element.* There is a real number 1 with the property that

$$x.1 = x$$

   for all $x \in \mathbb{R}$. Furthermore, $1 \neq 0$.

8. Every non-zero real number has a *multiplicative inverse.* Given any real number $x \neq 0$, there is a real number $\frac{1}{x}$ so that

$$x\frac{1}{x} = 1$$

9. Multiplication *distributes* over addition

$$x(y + z) = xy + xz$$

   for all $x, y, z \in \mathbb{R}$.

Q1]. . . [20 points] Consider the following statement about integers a.
Statement P: *If* $4 \mid a$, *then* $4 \mid a^2$.

1. Is statement P true or false? TRUE
   Please support your answer with a proof or counterexample.

   <u>Proof</u> By hypothesis $4 | a$. This means $a = 4k$ for some $k \in \mathbb{Z}$.
   Thus $a^2 = (4k)^2 = 4(4k^2)$
   Now $4k^2 \in \mathbb{Z}$ & so by definition $4 | a^2$. $\boxtimes$

2. Write down the converse of statement P.

   If $4 | a^2$, then $4 | a$.

3. Is the converse of P true or false? FALSE
   Please support your answer with a proof or counterexample.

   <u>Counterexample</u>. Let $a = 2$
   Then $a^2 = 4$ & $4 | 4 \Rightarrow 4 | a^2$
   but $4 \nmid 2 \Rightarrow 4 \nmid a$.

4. Write down the contrapositive of P.

   If $4 \nmid a^2$, then $4 \nmid a$.

5. Is the contrapositive of P true or false? TRUE
   Please support your answer with a proof or counterexample.

   This is true because the contrapositive is logically equivalent to the original statement, & we have already given a proof of the original statement in part 1 above.

Q2]... [20 points] Write down a truth table for the following compound statement

$$(\neg P) \wedge Q \wedge (\neg R)$$

P	Q	R	$\neg P$	$\neg R$	$(\neg P) \wedge Q \wedge (\neg R)$
T	T	T	F	F	F
T	T	F	F	T	F
T	F	T	F	F	F
T	F	F	F	T	F
F	T	T	T	F	F
F	T	F	T	T	T
F	F	T	T	F	F
F	F	F	T	T	F

Write down the negation of the statement above in a form that does not use the symbol $\wedge$.

$$\neg(\neg P \wedge Q \wedge \neg R) \equiv \neg(\neg P) \vee \neg Q \vee \neg(\neg R) \quad \text{--- by de Morgan}$$

$$\equiv \boxed{P \vee (\neg Q) \vee R} \quad \text{---. by } \neg\neg X \equiv X$$

Q3]... [20 points] Give a careful proof of the following proposition about real numbers x and y. If it helps, you may use the list of properties on the attached page and you may use the fact (proven in class) that the product of an arbitrary real number and 0 is equal to 0.

If $x \neq 0$ and $y \neq 0$, then $xy \neq 0$.

Proof We will prove this statement by proving its contrapositive; namely,

If $xy = 0$, then $x = 0$ OR $y = 0$.

This latter statement is logically equivalent to the following statement:

If $xy = 0$ and $x \neq 0$, then $y = 0$. —(*)

This is because $P \to (Q \vee R)$ is logically equivalent to $(P \wedge \neg Q) \to R$.

So we shall prove the original statement by proving the logically equivalent reformulation, (*).

Now $x \neq 0$ --- hypothesis of (*)

Therefore $\exists \frac{1}{x} \in \mathbb{R}$ such that $x \frac{1}{x} = 1$ --- Property 8 on our list.

Thus $y = y \cdot 1$ ---- property 7 on our list.

$= y(x \frac{1}{x})$ ---- substitution for 1 using the previous equality.

$= (yx)\frac{1}{x}$ ---- property 6 on our list

$= (xy)\frac{1}{x}$ ---- property 5 on our list

$= 0 \cdot \frac{1}{x}$ ---- the other hypotheses of (*)

$= 0$ ---- we are told that we can use the result: $(\forall a \in \mathbb{R})(0 \cdot a = 0)$.

Therefore $y = 0$ and (*) is proven. $\boxtimes$

Q4]... [20 points] Let $P(x, y, z)$ be the predicate $x + y = z$. Say which of the following quantified statements are true for the universal set $\mathbb{Z}$ of all integers. Give reasons to support your answers.

1. $(\exists x \in \mathbb{Z})(\exists y \in \mathbb{Z})(\exists z \in \mathbb{Z})P(x, y, z)$

<u>TRUE</u>. The <u>example</u> $x=1,\ y=2,\ z=3$ $(1+2=3)$ establishes the truth of this statement.

2. $(\forall x \in \mathbb{Z})(\forall y \in \mathbb{Z})(\forall z \in \mathbb{Z})P(x, y, z)$

<u>FALSE</u> The <u>counterexample</u> $x=1,\ y=2,\ z=10$ $(1+2 \neq 10)$ shows that this statement is false.

3. $(\forall x \in \mathbb{Z})(\forall y \in \mathbb{Z})(\exists z \in \mathbb{Z})P(x, y, z)$

<u>TRUE</u> Given $x, y \in \mathbb{Z}$, the closure of $\mathbb{Z}$ under addition means $x+y \in \mathbb{Z}$

<u>Pick</u> $z = x+y$

This <u>choice</u> of z proves that the statement is true.

4. $(\exists z \in \mathbb{Z})(\forall x \in \mathbb{Z})(\forall y \in \mathbb{Z})P(x, y, z)$

<u>FALSE</u> It's negation is the statement

$$(\forall z \in \mathbb{Z})(\exists x \in \mathbb{Z})(\exists y \in \mathbb{Z})(x+y \neq z)$$

Given any $z \in \mathbb{Z}$, the <u>choice</u> $x = z$ and $y = 3$ $(z+3 \neq z)$ shows that the negation is true.

Q5]... **[20 points]** Give a proof of the following statement about integers a:

$$\text{If } 11 \nmid a, \text{ then } 11 \nmid a^2$$

If it helps to speed up the proof, you can use congruences and their properties.

Proof By hypothesis $11 \nmid a$. This means $a \not\equiv 0 \bmod 11$. Thus

$a \equiv 1 \bmod 11$ OR $a \equiv 2 \bmod 11$ OR --- OR $a \equiv 10 \bmod 11$.

We consider all 10 cases in the following table.

a	$\bmod 11$	$a^2 \bmod 11$
1	$10 \equiv -1$	$(\pm 1)^2 \equiv 1$
2	$9 \equiv -2$	$(\pm 2)^2 \equiv 4$
3	$8 \equiv -3$	$(\pm 3)^2 \equiv 9$
4	$7 \equiv -4$	$(\pm 4)^2 \equiv 16 \equiv 5$
5	$6 \equiv -5$	$(\pm 5)^2 \equiv 25 \equiv 3$

Note that in all 10 cases $a^2 \not\equiv 0 \bmod 11$.

Thus $11 \nmid a^2$. $\boxed{\checkmark}$

Use a divisibility test to check if the number 12, 345, 678 divisible by 9? Give a clear statement of the test that you are using.

Test "The integer $a_n \cdots a_1 a_0$ written in base 10 is divisible by 9 if and only if the sum of its digits $a_n + \cdots + a_0$ is divisible by 9."

$$9 \mid 12{,}345{,}678 \iff 9 \mid (1+2+\cdots+8)$$

$$\iff 9 \mid 36$$

which is true, $9(4) = 36$.

Therefore $9 \mid 12{,}345{,}678$.