Some people gave nice proofs by contradiction in Q3.

<u>To prove</u>: $(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})\ (x \neq 0) \wedge (y \neq 0) \longrightarrow (xy \neq 0)$.

<u>Proof</u>: We argue by contradiction.

[+] Assume to the contrary that the <u>negation</u> of the statement above is <u>true</u>. This means

$$(*)\quad (\exists x \in \mathbb{R})(\exists y \in \mathbb{R})\ \underbrace{(x \neq 0)}_{hyp_1} \wedge \underbrace{(y \neq 0)}_{hyp_2} \wedge \underbrace{(xy = 0)}_{hyp_3}.$$

[This is "nice" b/c it gives us 3 hypotheses to work with].

By hyp_1 $\quad x \neq 0 \Rightarrow \exists \frac{1}{x} \in \mathbb{R}$ such that $\frac{1}{x}x = 1$ --- prop (8).

Thus
$$\begin{aligned} y &= 1.y && \text{--- prop (7)} \\ &= (\tfrac{1}{x}x)y && \text{--- subst}^{\underline{n}} \\ &= \tfrac{1}{x}(xy) && \text{--- prop (6)} \\ &= \tfrac{1}{x}(0) && \text{--- } hyp_3 \\ &= 0 && \text{--- we proved } \boxed{a.0 = 0} \text{ in class.} \end{aligned}$$

$\Rightarrow \quad y = 0$

But $hyp_2 \Rightarrow \quad y \neq 0$

These 2 statements form a contradiction.

Therefore our assumption [+] is wrong, and so the original statement is true. $\blacksquare$

There was a variation --- but still starting from the framework of a proof by contradiction & the 3 hypotheses ---.

By hyp_1 $\quad x \neq 0 \Rightarrow \exists \frac{1}{x} \in \mathbb{R}$ such that $\frac{1}{x}x = 1$ --- prop(8)

By hyp_2 $\quad y \neq 0 \Rightarrow \exists \frac{1}{y} \in \mathbb{R}$ such that $\frac{1}{y}y = 1$ --- prop(8)

$$\begin{aligned}
\Rightarrow \; 1 &= 1.1 && \text{--- prop (7)} \\
&= \left(\tfrac{1}{x}x\right)\left(\tfrac{1}{y}y\right) && \text{--- subst}^{=} \\
&= \left(\tfrac{1}{x}\tfrac{1}{y}\right)(xy) && \text{---- several usages of prop(5),(6)} \\
&= \left(\tfrac{1}{x}\tfrac{1}{y}\right)0 && \text{---- } hyp_3 \\
&= 0 && \text{--- proven in class } \boxed{a.0 = 0}
\end{aligned}$$

$$\left.\begin{array}{ll} \text{Thus} & 1 = 0 \\ \text{But} & 1 \neq 0 \end{array}\right] \text{--- in prop(7) on sheet}$$

These 2 statements form a Contradiction

⋮

etc --

⋮

Remark: All 3 proofs (2 proofs by contradiction above + the proof in the solution handout) use ~~the~~ Property (8)

$$\boxed{\text{If } x \neq 0 \text{ then } \exists \tfrac{1}{x} \text{ so that } \tfrac{1}{x}x = 1.}$$

This is the <u>essential ingredient</u> in these proofs.

Why?

Because the proofs would not work, <u>and</u> (more importantly) the statement of the theorem would be "wrong" in an arithmetic system where $\frac{1}{x}$ did not exist.

Counterexample to the statement of the theorem.

$\Rightarrow$ e.g. $(\{0,1,2,3,4,5\}, +_6, \times_6)$

is a system of arithmetic that satisfies properties (1), (2), (3), (4), (5), (6), (7) and $1 \neq 0$, (9) but <u>NOT</u> (8).

eg $2 \neq 0$ but

$$\left\{\begin{array}{lll} 2(0) \equiv 0 \not\equiv 1 & \text{mod } 6 \\ 2(1) \equiv 2 \not\equiv 1 & \text{mod } 6 \\ 2(2) \equiv 4 \not\equiv 1 & \text{mod } 6 \\ 2(3) \equiv 0 \not\equiv 1 & \text{mod } 6 \\ 2(4) \equiv 8 \equiv 2 \not\equiv 1 & \text{mod } 6 \\ 2(5) \equiv 10 \equiv 4 \not\equiv 1 & \text{mod } 6 \end{array}\right.$$

ie. $\nexists$ "$x = \frac{1}{2}$" in $\{0,1,\ldots,5\}$ so that $\boxed{x \cdot 2 = 1}$.

<u>Also</u>

$$\boxed{\begin{array}{l} (2)(3) \equiv 6 \equiv 0 \pmod 6 \\ \text{yet } 2 \not\equiv 0 \text{ mod } 6 \\ \text{and } 3 \not\equiv 0 \text{ mod } 6 \end{array}}$$