Functions, images and pre-images.

1. **Definition (Image).** Let $f : A \to B$ be a function and let $U \subset A$. The *image* of U under f, denoted by $f(U)$, is defined by

$$f(U) \;=\; \{f(a) \mid a \in U\}.$$

   It is a subset of B. In the special case $U = A$ we call its image $f(A)$ the *range* of the function f.

2. **Definition (Pre-image).** Let $f : A \to B$ be a function and let $V \subset B$. The *pre-image* of V under f, denoted by $f^{-1}(V)$, is defined by

$$f^{-1}(V) \;=\; \{a \in A \mid f(a) \in V\}.$$

   It is a subset of A. In the special case $V = \{b\}$ is a singleton set consisting of one element of B we denote the pre-image $f^{-1}(\{b\})$ by $f^{-1}(b)$ for ease of notation.

3. **Properties of images.** Prove the following properties.

   (a) $f(\emptyset) = \emptyset$.

   (b) If $U_1 \subset U_2$, then $f(U_1) \subset f(U_2)$.

   (c) $f(U_1 \cup U_2) = f(U_1) \cup f(U_2)$.

   (d) $f(U_1 \cap U_2) \subset f(U_1) \cap f(U_2)$. Give an example to show that these sets need not be equal.

4. **Properties of pre-images.** Prove the following properties.

   (a) If $V \subset B$ is such that $f(A) \cap V = \emptyset$, then $f^{-1}(V) = \emptyset$.

   (b) If $V_1 \subset V_2$, then $f^{-1}(V_1) \subset f^{-1}(V_2)$.

   (c) $f^{-1}(V_1 \cup V_2) = f^{-1}(V_1) \cup f^{-1}(V_2)$.

   (d) $f^{-1}(V_1 \cap V_2) = f^{-1}(V_1) \cap f^{-1}(V_2)$.

5. **Miscellaneous properties.** Prove the following properties.

   (a) $U \subset f^{-1}(f(U))$. Give an example to show that these sets need not be equal.

   (b) $f(f^{-1}(V)) \subset V$. Give an example to show that these sets need not be equal.