Theorem. If an integer a is odd, then a^3 is odd.

Two column format.

Statement	**Reason**
a is odd	hypothesis
$a = 2p + 1$ for some $p \in \mathbb{Z}$	definition of odd
$a^3 = (2p+1)^3 = 8p^3 + 12p^2 + 6p + 1$	algebra
$a^3 = 2(4p^3 + 6p^2 + 3p) + 1$	algebra
But $4p^3 + 6p^2 + 3p \in \mathbb{Z}$	closure of $\mathbb{Z}$ under $+$ and $\times$
Thus a^3 is odd	definition of odd

Paragraph format

By hypothesis a is odd. This means $a = 2p + 1$ for some $p \in \mathbb{Z}$. Therefore

$$a^3 = (2p+1)^3 = 8p^3 + 12p^2 + 6p + 1 = 2(4p^3 + 6p^2 + 3p) + 1.$$

Thus $a^3 = 2q + 1$ where $q = (4p^3 + 6p^2 + 3p) \in \mathbb{Z}$ by closure of $\mathbb{Z}$ under addition and multiplication, and so a^3 is odd. $\square$

Theorem. If an integer a is even, then a^3 is even.

Two column format.

Statement	**Reason**
a is even	hypothesis
$a = 2p$ for some $p \in \mathbb{Z}$	definition of even
$a^3 = (2p)^3 = 8p^3$	algebra
$a^3 = 2(4p^3)$	algebra
But $4p^3 \in \mathbb{Z}$	closure of $\mathbb{Z}$ under $\times$
Thus a^3 is even	definition of even

Paragraph format

By hypothesis a is even. This means $a = 2p$ for some $p \in \mathbb{Z}$. Therefore

$$a^3 = (2p)^3 = 8p^3 = 2(4p^3).$$

Thus $a^3 = 2q$ where $q = (4p^3) \in \mathbb{Z}$ by closure of $\mathbb{Z}$ under multiplication, and so a^3 is even. $\square$

Theorem. If a is an even integer and b is an even integer, then

(i) ab is an even integer, and

(ii) $a+b$ is an even integer.

Proof. By hypothesis $a=2p$ for some $p \in \mathbb{Z}$ and $b=2q$ for some $q \in \mathbb{Z}$. Thus

$$ab=(2p)(2q)=2(2pq)$$

is even since $2pq \in \mathbb{Z}$ by closure properties. Thus conclusion (i) holds.

Also

$$a+b=(2p)+(2q)=2(p+q)$$

is even since $p+q \in \mathbb{Z}$ by closure properties. Thus conclusion (ii) holds. □

Theorem. If a is an even integer and b is an odd integer, then

(i) ab is an even integer, and

(ii) $a+b$ is an odd integer.

Proof. By hypothesis $a=2p$ for some $p \in \mathbb{Z}$ and $b=2q+1$ for some $q \in \mathbb{Z}$. Thus

$$ab=(2p)(2q+1)=2(p(2q+1))$$

is even since $p(2q+1) \in \mathbb{Z}$ by closure properties. Thus conclusion (i) holds.

Also

$$a+b=(2p)+(2q+1)=2(p+q)+1$$

is odd since $p+q \in \mathbb{Z}$ by closure properties. Thus conclusion (ii) holds. □

Theorem. If a is an odd integer and b is an odd integer, then

(i) ab is an odd integer, and

(ii) $a+b$ is an even integer.

Proof. By hypothesis $a=2p+1$ for some $p \in \mathbb{Z}$ and $b=2q+1$ for some $q \in \mathbb{Z}$. Thus

$$ab=(2p+1)(2q+1)=4pq+2p+2q+1=2(2pq+p+q)+1$$

is odd since $2pq+p+q \in \mathbb{Z}$ by closure properties. Thus conclusion (i) holds.

Also

$$a+b=(2p+1)+(2q+1)=2(p+q+1)$$

is even since $p+q+1 \in \mathbb{Z}$ by closure properties. Thus conclusion (ii) holds. □