The "Die Hard" Problems

Question. Is it possible to obtain exactly 4 gal. of water using only 3 gal. and 5 gal. containers? If so, show how it is possible. If not, show why not.

Answer. Yes it is possible. Here is one solution.

Activity	**3 gal can**	**5 gal can**
Start	0	0
Fill 5 gal	0	5
Pour 5 $\to$ 3	3	2
Pour 3 $\to$ ground	0	2
Pour 5 $\to$ 3	2	0
Fill 5 gal	2	5
Pour 5 $\to$ 3	3	4

Note that the volume in any can at any time is an *integer linear combination* of 3 and 5. Here are the values expressed explicitly as combinations.

Activity	**3 gal can**	**5 gal can**
Start	$0 = (0)3 + (0)5$	$0 = (0)3 + (0)5$
Fill 5 gal	$0 = (0)3 + (0)5$	$5 = (0)3 + (1)5$
Pour 5 $\to$ 3	$3 = (1)3 + (0)5$	$2 = (-1)3 + (1)5$
Pour 3 $\to$ ground	$0 = (0)3 + (0)5$	$2 = (-1)3 + (1)5$
Pour 5 $\to$ 3	$2 = (-1)3 + (1)5$	$0 = (0)3 + (0)5$
Fill 5 gal	$2 = (-1)3 + (1)5$	$5 = (0)3 + (1)5$
Pour 5 $\to$ 3	$3 = (1)3 + (0)5$	$4 = (-2)3 + (2)5$

The last combination is obtained as follows. Note that we just have to "top off" the 3 gal container by adding $(1)3 - ((-1)3 + (1)5) = (2)3 + (-1)5$ gallons to it from the 5 gal container. This leaves $(1)5 - ((2)3 + (-1)5) = (2)5 + (-2)3$ gallons in the 5 gal container at the very end.

Question. Is it possible to obtain exactly 8 gal. of water using only 6 gal. and 21 gal. containers? If so, show how it is possible. If not, show why not.

Answer. No. The reason is that, as in the previous example, the volumes in each container at any time are always *integer linear combinations* of 6 and 21. However, in this case $3 \mid 6$ and $3 \mid 21$. Therefore

$$3 \mid (x)6 + (y)21$$

for any integers x and y. Thus all volumes that we obtain are multiples of 3. But 8 is not a multiple of 3, and so we can never obtain 8 gallons.

Remarks. There are a few important observations that you can take from this example.

1. The volume of water that you have in any can at any stage is always an integer linear combination of the two can capacity numbers, a and b.

2. Any large common divisor of a and b (in particular $\gcd(a,b)$) serves as an *obstruction* to obtaining certain volumes (e.g., $3 \nmid 8$).

3. When first trying to solve these problems you were probably trying to obtain as small a (positive) number as possible as an integer linear combination of a and b.

The take away message is that there is some connection between $\gcd(a,b)$ and smallest positive, integer linear combinations of a and b. The statement and proof of the following result should not be a surprise.

Theorem (Bézout). Let a and b be integers, not both zero. Then there exists $l, m \in \mathbb{Z}$ such that

$$\gcd(a,b) \;=\; la + mb.$$

The proof is contained in the Least Principle handout.