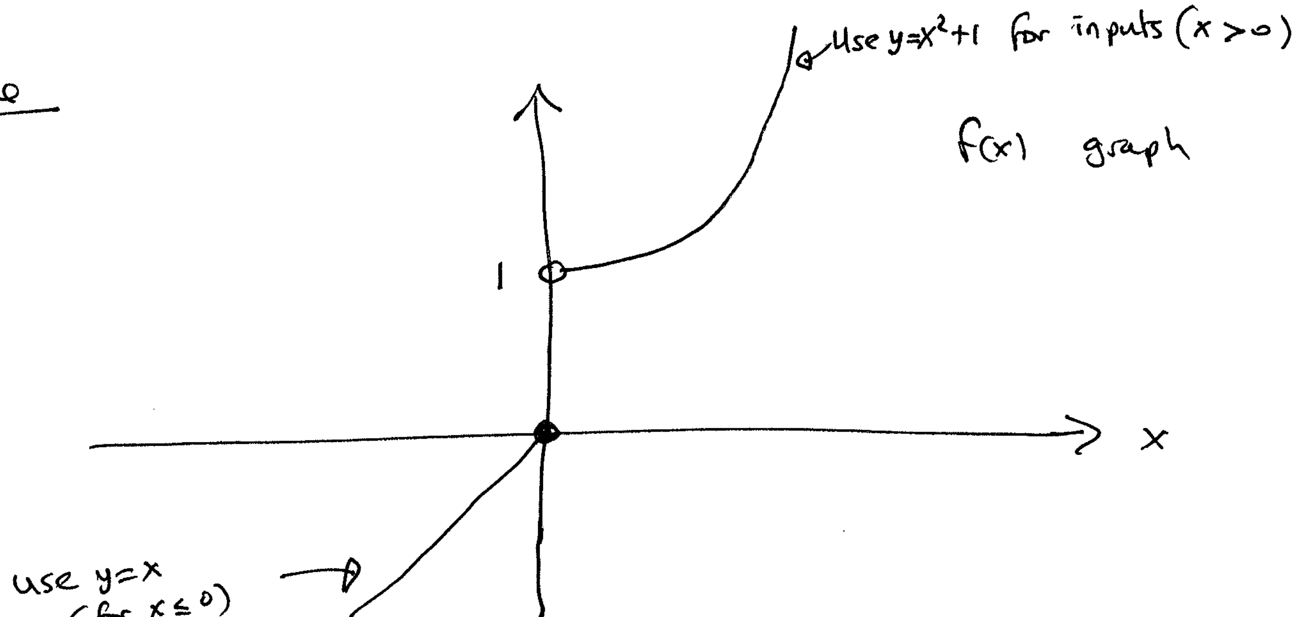
$f(x)$ is <u>continuous at a</u> means

$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in dom(f))\left(0 < |x-a| < \delta \longrightarrow |f(x)-f(a)| < \epsilon\right)$$

Prove that $f(x) = \begin{cases} x^2+1 & \text{if } x > 0 \\ x & \text{if } x \le 0 \end{cases}$

is <u>NOT continuous</u> at 0.

<u>Picture</u>



It is <u>intuitively clear</u> from the picture of the graph of $f(x)$ that $f(x)$ is not continuous at 0. Indeed it looks like there is a "jump of size 1" as we move inputs from 0 to just to right of 0.

How to formalize this . . .

Step 1 Write down the definition of what it means for $f(x)$ to be NOT continuous at a.

$f(x)$ is NOT continuous at a means

$$\neg\big((\forall \varepsilon>0)(\exists \delta>0)(\forall x \in \mathrm{dom}(f))\big)\big(0<|x-a|<\delta \longrightarrow |f(x)-f(a)|<\varepsilon\big)$$

$$\equiv (\exists \varepsilon>0)\,\neg(\exists \delta>0)(\forall x \in \mathrm{dom}(f))\big(0<|x-a|<\delta \rightarrow |f(x)-f(a)|<\varepsilon\big)$$

$$\equiv (\exists \varepsilon>0)(\forall \delta>0)\,\neg(\forall x \in \mathrm{dom}(f))\big(0<|x-a|<\delta \rightarrow |f(x)-f(a)|<\varepsilon\big)$$

$$\equiv (\exists \varepsilon>0)(\forall \delta>0)(\exists x \in \mathrm{dom}(f))\,\neg\big(0<|x-a|<\delta \rightarrow |f(x)-f(a)|<\varepsilon\big)$$

$$\equiv (\exists \varepsilon>0)(\forall \delta>0)(\exists x \in \mathrm{dom}(f))\big(0<|x-a|<\delta \wedge \neg(|f(x)-f(a)|<\varepsilon)\big)$$

$$\equiv (\exists \varepsilon>0)(\forall \delta>0)(\exists x \in \mathrm{dom}(f))\big(0<|x-a|<\delta \wedge |f(x)-f(a)|\geq\varepsilon\big)$$

where the 2nd last equivalence was really just negation of a conditional:

$$\neg(P \longrightarrow Q) \equiv P \wedge \neg Q$$

Step ② In order to show $f(x)$ is not continuous at 0 ③

we must show

$$(\exists \varepsilon > 0)\ (\forall \delta > 0)\ (\exists x \in dom(f))\ \left(0 < |x-a| < \delta \ \wedge \ |f(x) - f(a)| \geq \varepsilon\right)$$

Now $a = 0$, $f(a) = 0$, $dom(f) = \mathbb{R}$

domain of f.

so we can rewrite our statement as follows.

$$(\exists \varepsilon > 0)\ (\forall \delta > 0)\ (\exists x \in \mathbb{R})\ \left(0 < |x| < \delta \ \wedge \ |f(x)| \geq \varepsilon\right). \quad (**)$$

We need to find two numbers (ε and x)...

(1) a global output tolerence $\varepsilon > 0$ which is small enough to detect a "jump of 1" in the graph.

eg. $\varepsilon = \frac{1}{2}$

(2) Given a choice of $\delta > 0$ we have to find an input x so that

(i) $0 < |x| < \delta$

and (ii) $|f(x)| \geq \varepsilon$.

We'll guarantee (ii) by choosing $x > 0$ (since outputs "jump by 1" as inputs increase to right of 0). Thus (i) becomes

$$0 < x < \delta$$

eg. $x = \frac{\delta}{10}$.

Step (3) Write things out carefully.

(1) Pick $\varepsilon = \frac{1}{2}$.

(2) Given any $\delta > 0$, pick $x = \frac{\delta}{10}$.

Then $0 < \frac{\delta}{10} < \delta$ & so $0 < |x| < \delta$ is satisfied

Also $|f(x)| = \left|f\left(\frac{\delta}{10}\right)\right| = \left|\left(\frac{\delta}{10}\right)^2 + 1\right|$

$$= \frac{\delta^2}{100} + 1$$

$$> 1$$

$$\geq \frac{1}{2} = \varepsilon$$

So $|f(x)| \geq \varepsilon$ is satisfied.

Thus statement $\binom{*}{*}$ is true.

& so $f(x)$ is NOT continuous at $x=0$.

Remark The first time one works with the formal definitions of limits + continuity can be daunting. As this example shows, it pays to work slowly through a specific example, switching from your intuition to rigorous statements.

eg:

"a jump of 1" $\rightsquigarrow$ pick $\varepsilon = \frac{1}{2}$

Outputs jump by 1 as we move to right of 0 $\rightsquigarrow$ pick $x > 0$

These were the two key choices in making this example work.

The rest eg. $x = \frac{\delta}{2}$ (or $x = \frac{\delta}{10}$) were forced on us by the inequalities we had to work with: $0 < |x| < \delta$. etc.

Key fact x had to be "expressed in terms of δ" because we are arguing the existence of x satisfying an inequality involving δ after δ is chosen.

In proving that $g(x) = 7x$ is continuous at 0, we need to verify a nested qualifier statement:

$$(\forall \varepsilon > 0)\ (\exists \delta > 0)\ (\forall x \in \mathbb{R})\ \left(0 < |x - 0| < \delta \longrightarrow |f(x) - f(0)| < \varepsilon\right).$$

Here we need to pick (assert the existence of) a $\delta > 0$ given a choice of $\varepsilon > 0$, so it makes sense to "express δ in terms of ε." An expression becomes clear after some rough work...

We want $|g(x) - g(0)| < \varepsilon$

i.e. $|7x - 0| < \varepsilon$

i.e. $7|x - 0| < \varepsilon$

i.e. $|x - 0| < \frac{\varepsilon}{7}$

so it makes sense to pick $\boxed{\delta = \frac{\varepsilon}{7}}$

(anything smaller would also work)

Here's the proof.

Given $\varepsilon > 0$, pick $\delta = \frac{\varepsilon}{7}$. Clearly $\delta > 0$, and for any $x \in \mathbb{R}$ satisfying $0 < |x - 0| < \delta$,

then $|x - 0| < \frac{\varepsilon}{7}$.

$\Rightarrow 7|x - 0| < 7\left(\frac{\varepsilon}{7}\right) = \varepsilon$

$\Rightarrow |7x - 7.0| < \varepsilon$

$\Rightarrow |7x - 0| < \varepsilon$

$\Rightarrow |g(x) - g(0)| < \varepsilon$.

Therefore, by definition, $g(x)$ is continuous at 0. ∎