Injective, Surjective, Bijective Functions.

1. **Definition (Injective).** A function $f : A \to B$ is said to be *injective* if

$$(\forall a_1 \in A)(\forall a_2 \in A)(f(a_1) = f(a_2) \;\longrightarrow\; a_1 = a_2)$$

2. **Definition (Surjective).** A function $f : A \to B$ is said to be *surjective* if

$$(\forall b \in B)(\exists a \in A)(f(a) = b)$$

3. **Definition (Bijective).** A function $f : A \to B$ is said to be *bijective* if f is both injective and surjective.

4. If $f : A \to B$ and $g : B \to C$ are both injective, then $g \circ f : A \to C$ is injective.

5. If $g \circ f : A \to C$ is injective, then $f : A \to B$ is injective, but $g : B \to C$ need not be.

6. If $f : A \to B$ and $g : B \to C$ are both surjective, then $g \circ f : A \to C$ is surjective.

7. If $g \circ f : A \to C$ is surjective, then $g : B \to C$ is surjective, but $f : A \to B$ need not be.

8. If $f : A \to B$ and $g : B \to C$ are both bijective, then $g \circ f : A \to C$ is bijective.

9. If $g \circ f : A \to C$ is bijective, then $f : a \to B$ is injective, and $g : B \to C$ is surjective, but f need not be surjective and g need not be injective.

10. If $f : A \to B$ is injective, then $f(U_1 \cap U_2) = f(U_1) \cap f(U_2)$ for all $U_1, U_2 \subseteq A$.

11. If $f : A \to B$ is injective, then $f^{-1}(f(U)) = U$ for all $U \subseteq A$.

12. If $f : A \to B$ is surjective, then $f(f^{-1}(V)) = V$ for all $V \subseteq B$.

13. The *identity function*

$$\mathbb{I}_A : A \to A : a \mapsto a$$

for all $a \in A$ is a bijection of A to itself.

14. If $f : A \to B$ is a bijection, then there is a well-defined *inverse function*

$$f^{-1} : B \to A$$

defined by $f^{-1}(b) = a$ where a is the unique element of A such that $f(a) = b$. Existence of a is guaranteed by surjectivity of f and uniqueness is guaranteed by injectivity of f. It "undoes" the function f in the following sense

$$f \circ f^{-1} \;=\; \mathbb{I}_A \qquad \text{and} \qquad f^{-1} \circ f \;=\; \mathbb{I}_A$$