Logic of the proof that Least Principle implies Principle of Induction.

In the Least Principle handout it is proven that the Least Principle (L) is equivalent to the Principle of Mathematical Induction (I). This is achieved by proving a circle of implications, one of which is

$$L \rightarrow I$$

Remember that the statement of induction, is itself a conditional

$$H \rightarrow C$$

where H is the hypothesis and C is the conclusion.

So when you are asked to prove that $L \rightarrow I$, you are really being asked to prove that the following is true

$$L \rightarrow (H \rightarrow C)$$

It can be easy to get confused by the nested conditionals; which hypothesis can I work with and what conclusion am I trying to prove? Don't despair! Just let the logic unwrap it for you. Remember that a conditional $A \rightarrow B$ is equivalent to $\neg A \vee B$.

$$L \rightarrow (H \rightarrow C) \equiv \neg L \vee (H \rightarrow C) \equiv \neg L \vee (\neg H \vee C) \equiv \neg(L \wedge H) \vee C \equiv (L \wedge H) \rightarrow C.$$

This means that proving $L \rightarrow I$ is the same as using the Least Principle L and the hypotheses H of Mathematical Induction together to prove the conclusion C of Mathematical Induction. Recall that there are two hypotheses in the statement of induction $H = H_1 \wedge H_2$ so you are really using three hypotheses, L, H_1 and H_2, and trying to prove the conclusion C.

Here they all are written out.
L: Every non-empty subset of $\mathbb{N}$ has a least element.
H_1: The statement $P(1)$ is true.
H_2: If the statement $P(k)$ is true, then the statement $P(k+1)$ is true.
C: The statement $P(n)$ is true for all positive integers $n \in \mathbb{N}$.

You are allowed to use L, H_1 and H_2 and you have to argue that C follows.

Comment. To add to the confusion about nested conditionals, notice that the second hypothesis of induction is itself a conditional statement (denote it by $S \rightarrow T$ for fun). What you are really proving when proving $L \rightarrow I$ is the following statement

$$L \rightarrow [(H_1 \wedge (S \rightarrow T)) \rightarrow C].$$

This seems scary, but the logical equivalences above tell us that all is good; it's just a disguised form of three hypotheses implies one conclusion:

$$(L \wedge H_1 \wedge (S \rightarrow T)) \rightarrow C.$$

Overleaf we provide a concrete example where the Least Principle may be used instead of Induction.

Example. For every natural number n

$$1 + 2 + \cdots + n \;=\; \frac{n(n+1)}{2}$$

Proof. We argue by contradiction. Suppose to the contrary that the statement

$$(\forall n \in \mathbb{N})(1 + 2 + \cdots + n \;=\; \frac{n(n+1)}{2})$$

is false. This means that its negation

$$(\exists n \in \mathbb{N})(1 + 2 + \cdots + n \;\neq\; \frac{n(n+1)}{2}) \qquad (*)$$

is true.

Let $S = \{m \in \mathbb{N} \mid 1 + 2 + \cdots + n \neq \frac{n(n+1)}{2}\}$. By $(*)$, the set S is non-empty. By the Least Principle S has a least element n_0 say.

Note that $n_0 \neq 1$. This is because $1 = \frac{1(1+1)}{2}$ is true, and so 1 does not belong to the set S. Therefore $n_0 \geq 2$ and so the number $n_0 - 1$ is a natural number. Since n_0 is the least element of S, we know that $n_0 - 1 \notin S$. Therefore

$$1 + 2 + \cdots + (n_0 - 1) \;=\; \frac{(n_0-1)((n_0-1)+1)}{2} \;=\; \frac{n_0(n_0-1)}{2}.$$

Adding n_0 to both sides of this equation gives

$$1 + 2 + \cdots + n_0 \;=\; \frac{n_0(n_0-1)}{2} + n_0 \;=\; \frac{n_0(n_0-1) + 2n_0}{2} \;=\; \frac{n_0(n_0+1)}{2}$$

and so the equation holds for n_0. But this contradicts the fact that n_0 belongs to S.

This contradiction arose from the assumption that the statement of the theorem is false. □

Remarks.

1. Although we never used the word "induction" anywhere in this proof, the details of what we had to do are *virtually identical* to the work done in a standard proof by Induction. In particular, we had to argue directly that the formula holds true for $n = 1$ (this is the base case of an induction proof) in order to say that $1 \notin S$ and thus that $n_0 \neq 1$. We also had to argue that the formula holding true for $(n_0 - 1)$ implies that the formula holds for n_0 (in order to obtain a contradiction), and this is identical to the $P(k) \to P(k+1)$ case of an induction proof (use the "dictionary" $k = n_0 - 1$ and $k + 1 = n_0$ to translate between the two).

2. The structure of the proof in this example can be used to give a proof by contradiction in any case where a proof by Induction would work. Indeed, this is the framework of the abstract proof that the Least Principle implies the Principle of Induction (given in the Least Principle handout).