Proofs using the Principle of Induction.

Recall the statement of the Principle of Induction in logical notation

$$(P(1) \wedge (\forall k \in \mathbb{N})(P(k) \rightarrow P(k+1))) \;\rightarrow\; (\forall n \in \mathbb{N})P(n)$$

and in friendlier terms.

Principle of Induction. Suppose $P(n)$ is a sentence about positive integers n.

$$\left.\begin{array}{l} \bullet \;\; P(1) \text{ true} \\ \bullet \;\; P(k) \text{ true } \rightarrow P(k+1) \text{ true} \end{array}\right\} \rightarrow P(n) \text{ true}, \forall n \in \mathbb{N}$$

There is the Principle of Strong Induction in logical notation

$$(P(1) \wedge (\forall k \in \mathbb{N})(P(1) \wedge \cdots \wedge P(k) \rightarrow P(k+1))) \;\rightarrow\; (\forall n \in \mathbb{N})P(n)$$

and in friendlier terms.

Principle of Strong Induction. Suppose $P(n)$ is a sentence about positive integers n.

$$\left.\begin{array}{l} \bullet \;\; P(1) \text{ true} \\ \bullet \;\; P(1 \,)\text{true} \wedge \cdots \wedge P(k) \text{ true } \rightarrow P(k+1) \text{ true} \end{array}\right\} \rightarrow P(n) \text{ true}, \forall n \in \mathbb{N}$$

Prove the following using either form of induction that appears to work best with the problem.

1. $$1 + \cdots + n \;=\; \frac{n(n+1)}{2}$$

2. $$1^2 + \cdots + n^2 \;=\; \frac{n(n+1)(2n+1)}{6}$$

3. $$1^3 + \cdots + n^3 \;=\; \left(\frac{n(n+1)}{2}\right)^2$$

4. $$\sum_{i=1}^{n}(2i-1) \;=\; n^2$$

5. $$a \equiv x \mod m \;\rightarrow\; a^n \equiv x^n \mod m$$

6. $$5 \mid (8^n - 3^n)$$

7. $$n! \;<\; n^n \qquad \text{for } n \geq 2$$

8. $$6 \mid n^3 - n$$

9. $$12 \mid (n^4 - n^2)$$

10. $$2^n \;>\; 2n \qquad\qquad \text{for every integer } n > 2$$

11. $$\frac{d^n e^{7x}}{dx^n} \;=\; 7^n e^{7x}$$

12. If n letters are put into an array of fewer than n pigeonholes, then some pigeonhole has at least two letters.

13. Let $d \in \mathbb{N}$.
$$(\forall n \in \mathbb{N})(\exists q \in \mathbb{Z})(\exists r \in \mathbb{Z})(n = qd + r) \wedge (0 \leq r < d)$$

14. Do a $-n = qd + r$ version of the above. This almost proves the DA by induction. What is left?

15. Every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.

16. Every integer greater than or equal to 2 can be expressed as a product of prime numbers.

17. $$\sum_{i=1}^{n} \frac{1}{i(i+1)} \;=\; \frac{n}{n+1}$$

18. $$n! > 2^n \qquad\qquad \text{for } n \geq 4$$

19. $$(\forall x \in \mathbb{R})(\forall n \in \mathbb{N})(x > -1) \;\rightarrow\; (1+x)^n \geq 1 + nx$$

20. The number of binary strings of length n is 2^n.

21. The number of lines in a truth table for a compound statement involving primitive statements $P_1, \ldots, P_n$ is 2^n.

22. $$\neg(P_1 \vee \cdots \vee P_n) \equiv \neg P_1 \wedge \cdots \wedge \neg P_n$$

23. $$\neg(P_1 \wedge \cdots \wedge P_n) \equiv \neg P_1 \vee \cdots \vee \neg P_n$$

24. $$P \wedge (Q_1 \vee \cdots \vee Q_n) \equiv (P \wedge Q_1) \vee \cdots \vee (P \wedge Q_n)$$

25. $$P \vee (Q_1 \wedge \cdots \wedge Q_n) \equiv (P \vee Q_1) \wedge \cdots \wedge (P \vee Q_n)$$

26. f_n is the nth Fibonacci number.
$$\sum_{i=1}^{n} f_i^2 \;=\; f_n f_{n+1}$$

27. f_n is the nth Fibonacci number.
$$f_n \;\geq\; (3/2)^{n-2}$$

The Fibonacci numbers are defined by
$$f_1 = 1, \quad f_2 = 1, \qquad f_n = f_{n-1} + f_{n-2} \quad (\forall n \geq 3)$$

Use the principle of strong induction to prove the following results.
Recall that the Fibonacci numbers are defined by

$$f_1 = 1, \quad f_2 = 1, \qquad f_n = f_{n-1} + f_{n-2} \quad (\forall n \geq 3)$$

1. Every postage amount of 18 cents or greater can be made up from 3 cent and 10 cent stamps.
2. $$12 \mid (n^4 - n^2)$$
3. f_n is the nth Fibonacci number.
$$f_n \;\geq\; (3/2)^{n-2}$$
4. f_n is the nth Fibonacci number.
$$f_n \;=\; \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}$$
5. Any two ways of bracketing a product of n numbers gives the same answer.

For the last problem, notice that this is about $n \geq 3$. For 3 numbers we have two ways of putting parentheses in to compute the product $a_1a_2a_3$, and they are equal by the *associativity property* of multiplication

$$a_1(a_2a_3) \;=\; (a_1a_2)a_3$$

For four numbers there are more bracketing possibilities to consider

$$((a_1a_2)a_3)a_4 \qquad (a_1a_2)(a_3a_4) \qquad a_1(a_2(a_3a_4)) \qquad (a_1(a_2a_3))a_4 \qquad a_1((a_2a_3)a_4)$$

But you can use applications of the basic associativity law for three numbers to show that all these expressions give the same answer. The problem asks you to prove that all of the ways of bracketing the product

$$a_1 \ldots a_n$$

and performing multiplications from the innermost bracketed numbers outwards give the same answer. Hint: It might be useful to define a *standard left-to-right bracketing* of $a_1 \ldots a_n$ to be

$$(\cdots((a_1a_2)a_3)a_4 \cdots a_{n-1})a_n$$

and then to prove that every bracketing of $a_1 \ldots a_n$ gives the same answer as the standard left-to-right bracketing.