Applications of congruences and the Division Algorithm

Give Proofs of the 4 propositions below. Use the notation and properties of congruences to write short proofs. Explicitly refer to the Division Algorithm to generate cases in each proof.

Prop. 1. If n is an integer, then $3 \mid (n^3 - n)$.

Prop. 2. If n is an integer, then $5 \mid (n^5 - n)$.

Prop. 3. If n is an integer, then $7 \mid (n^7 - n)$.

Prop. 4. If n is an integer, then $11 \mid (n^{11} - n)$.

Answer the following 5 questions. If you find counterexamples, please describe **all possible** counterexamples in each case.

Q1. Does 4 divide $n^4 - n$ for every integer n?

Q2. Does 6 divide $n^6 - n$ for every integer n?

Q3. Does 8 divide $n^8 - n$ for every integer n?

Q4. Does 9 divide $n^9 - n$ for every integer n?

Q5. Does 10 divide $n^{10} - n$ for every integer n?

Finally, make a conjecture about which integers a have the property that $a \mid (n^a - n)$ for all integers n.