Orthogonal Curvilinear Coordinates in 3–Dimensions

1. Consider a coordinate system in $\mathbb{R}^3$ defined by

$$\mathbf{r}(u_1, u_2, u_3) \;=\; \langle x(u_1, u_2, u_3), y(u_1, u_2, u_3), z(u_1, u_2, u_3)\rangle$$

Setting two of the coordinates u_i to be constant gives a parametric *coordinate curve* with the third coordinate as parameter. Since these coordinate curves are not usually straight lines, the coordinates are called *curvilinear*.

2. Taking partial derivatives gives *tangent vectors*

$$\frac{\partial \mathbf{r}}{\partial u_i} \;=\; \left\langle \frac{\partial x}{\partial u_i}, \frac{\partial y}{\partial u_i}, \frac{\partial z}{\partial u_i} \right\rangle$$

to the coordinate curves. The coordinates are said to be *orthogonal* if these three tangent vectors are mutually perpendicular (orthogonal) at each point of space

$$\frac{\partial \mathbf{r}}{\partial u_i} \cdot \frac{\partial \mathbf{r}}{\partial u_j} \;=\; 0 \text{ for } i \neq j$$

3. **Unit basis vectors.** One scales the tangent vectors to have length 1, to get a basis (or moving frame) of vectors at each point. Define the *scale factors* h_i by

$$h_i \;=\; \left| \frac{\partial \mathbf{r}}{\partial u_i} \right|$$

and define the unit vectors $\hat{\mathbf{u}}_i$ by

$$\hat{\mathbf{u}}_i \;=\; \frac{1}{h_i} \frac{\partial \mathbf{r}}{\partial u_i}$$

We usually order the basis vectors so that they form a right handed system: $\hat{\mathbf{u}}_1 \times \hat{\mathbf{u}}_2 = \hat{\mathbf{u}}_3$, $\hat{\mathbf{u}}_2 \times \hat{\mathbf{u}}_3 = \hat{\mathbf{u}}_1$, and $\hat{\mathbf{u}}_3 \times \hat{\mathbf{u}}_1 = \hat{\mathbf{u}}_2$.

4. **Cylindrical Coordinates.** Cylindrical coordinates are a example of an orthogonal curvilinear coordinate system.

$$\mathbf{r}(r, \theta, z) \;=\; \langle r\cos\theta, r\sin\theta, z\rangle$$

with tangent vectors, scale factors, and unit vectors given by

$$\frac{\partial \mathbf{r}}{\partial r} = \langle \cos\theta, \sin\theta, 0\rangle \qquad h_1 = 1 \qquad \hat{\mathbf{r}} = \langle \cos\theta, \sin\theta, 0\rangle$$

$$\frac{\partial \mathbf{r}}{\partial \theta} = \langle -r\sin\theta, r\cos\theta, 0\rangle \qquad h_2 = r \qquad \hat{\theta} = \langle -\sin\theta, \cos\theta, 0\rangle$$

$$\frac{\partial \mathbf{r}}{\partial z} = \langle 0, 0, 1\rangle \qquad h_3 = 1 \qquad \hat{\mathbf{z}} = \langle 0, 0, 1\rangle$$

You should verify that these are mutually orthogonal unit vectors.

5. **Spherical Coordinates.** Spherical coordinates are another example of an orthogonal curvilinear coordinate system.

$$\mathbf{r}(\rho, \phi, \theta) \;=\; \langle \rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi\rangle$$

with tangent vectors, scale factors, and unit vectors given by

$$\frac{\partial \mathbf{r}}{\partial \rho} = \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi\rangle \qquad h_1 = 1 \qquad \hat{\rho} = \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi\rangle$$

$$\frac{\partial \mathbf{r}}{\partial \phi} = \langle \rho\cos\phi\cos\theta, \rho\cos\phi\sin\theta, -\rho\sin\phi\rangle \qquad h_2 = \rho \qquad \hat{\phi} = \langle \cos\phi\cos\theta, \cos\phi\sin\theta, -\sin\phi\rangle$$

$$\frac{\partial \mathbf{r}}{\partial \theta} = \langle -\rho\sin\phi\sin\theta, \rho\sin\phi\cos\theta, 0\rangle \qquad h_3 = \rho\sin\phi \qquad \hat{\theta} = \langle -\sin\theta, \cos\theta, 0\rangle$$

You should verify that these are mutually orthogonal unit vectors.

6. **Gradient in Curvilinear Coordinates.** Let f be a scalar field (function). Recall that the components of a vector with respect to the usual basis $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ are simply the projections of the vector onto the $\hat{\mathbf{i}}, \hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ directions. Likewise, we compute the $\hat{\mathbf{u}}_i$–components of ∇f by projecting the vector ∇f onto $\hat{\mathbf{u}}_i$. This is

$$\nabla f \cdot \hat{\mathbf{u}}_i = \frac{1}{h_i}\frac{\partial f}{\partial x}\frac{\partial x}{\partial u_i} + \frac{1}{h_i}\frac{\partial f}{\partial y}\frac{\partial y}{\partial u_i} + \frac{1}{h_i}\frac{\partial f}{\partial z}\frac{\partial z}{\partial u_i} = \frac{1}{h_i}\frac{\partial f}{\partial u_i}$$

The first equality comes from the definition of ∇f and $\hat{\mathbf{u}}_i$, and the second inequality is just the chain rule.

Thus, we get the following formula for ∇f

$$\nabla f = \frac{1}{h_1}\frac{\partial f}{\partial u_1}\hat{\mathbf{u}}_1 + \frac{1}{h_2}\frac{\partial f}{\partial u_2}\hat{\mathbf{u}}_2 + \frac{1}{h_3}\frac{\partial f}{\partial u_3}\hat{\mathbf{u}}_3 \tag{1}$$

In **cylindrical coordinates** the gradient is

$$\nabla f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}$$

In **spherical coordinates** the gradient is

$$\nabla f = \frac{\partial f}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\hat{\phi} + \frac{1}{\rho\sin\phi}\frac{\partial f}{\partial \theta}\hat{\theta}$$

7. **Gradient Expression for the Basis Vectors.** Note that

$$\nabla u_i = \frac{1}{h_1}\frac{\partial u_i}{\partial u_1}\hat{\mathbf{u}}_1 + \frac{1}{h_2}\frac{\partial u_i}{\partial u_2}\hat{\mathbf{u}}_2 + \frac{1}{h_3}\frac{\partial u_i}{\partial u_3}\hat{\mathbf{u}}_3 = \frac{1}{h_i}\hat{\mathbf{u}}_i \tag{2}$$

The last equality holds because $\frac{\partial u_i}{\partial u_j}$ is equal to 0 when $j \neq i$, and is equal to 1 when $j = i$.

Thus, we have another expression for the $\hat{\mathbf{u}}_i$ in terms of gradients

$$\hat{\mathbf{u}}_1 = h_1 \nabla u_1 \qquad \hat{\mathbf{u}}_2 = h_2 \nabla u_2 \qquad \hat{\mathbf{u}}_3 = h_3 \nabla u_3 \tag{3}$$

This should make intuitive sense to you. Recall that ∇u_1 at a point P is normal to the level surface $u_1 = C$, a constant. The coordinate curves for u_2 and for u_3 through the point P by definition will keep u_1 fixed, and so they lie in the level surface $u_1 = C$. Therefore the normal vector ∇u_1 at P is perpendicular to the (scaled) tangent vectors $\hat{\mathbf{u}}_2$ and $\hat{\mathbf{u}}_3$ at P. So ∇u_1 is a multiple of $\hat{\mathbf{u}}_1$. Equation (2) tells us the precise multiple. Likewise for ∇u_2 and ∇u_3.

8. **The Divergence in Curvilinear Coordinates.** Let $\mathbf{F}$ be a vector field with coordinate functions F_i with respect to the unit vectors $\hat{\mathbf{u}}_i$. That is

$$\mathbf{F} \;=\; F_1\hat{\mathbf{u}}_1 \,+\, F_2\hat{\mathbf{u}}_2 \,+\, F_3\hat{\mathbf{u}}_3$$

where the F_i are functions of (u_1, u_2, u_3).

We compute the divergence $\nabla \cdot \mathbf{F}$ using properties of the differential operator ∇. First ∇ satisfies a sum rule, and so it suffices to determine each $\nabla \cdot (F_i\hat{\mathbf{u}}_i)$ individually. Furthermore, the product rule for $\nabla\cdot$ gives

$$\nabla \cdot (F_i\hat{\mathbf{u}}_i) \;=\; (\nabla F_i) \cdot \hat{\mathbf{u}}_i \,+\, F_i(\nabla \cdot \hat{\mathbf{u}}_i) \tag{4}$$

The first term on the right hand side of equation (4) is easy to compute now that we know an expression (from equation (1)) for the gradient. It is just

$$(\nabla F_i) \cdot \hat{\mathbf{u}}_i \;=\; \frac{1}{h_i}\frac{\partial F_i}{\partial u_i} \tag{5}$$

The second term on the right hand side of equation (4) takes a little more thought. For concreteness, we compute $F_1(\nabla \cdot \hat{\mathbf{u}}_1)$. The other cases $(i = 2, 3)$ are handled similarly.

$$\begin{aligned}
F_1(\nabla \cdot \hat{\mathbf{u}}_1) &= F_1\nabla \cdot (\hat{\mathbf{u}}_2 \times \hat{\mathbf{u}}_3) \\
&= F_1\nabla \cdot (h_2\nabla u_2 \times h_3\nabla u_3) \\
&= F_1\nabla \cdot (h_2h_3\nabla u_2 \times \nabla u_3) \\
&= F_1\nabla(h_2h_3) \cdot (\nabla u_2 \times \nabla u_3) \,+\, F_1h_2h_3\nabla \cdot (\nabla u_2 \times \nabla u_3) \\
&= F_1\nabla(h_2h_3) \cdot \frac{\hat{\mathbf{u}}_2 \times \hat{\mathbf{u}}_3}{h_2h_3} \,+\, 0 \\
&= F_1\nabla(h_2h_3) \cdot \frac{\hat{\mathbf{u}}_1}{h_2h_3} \\
&= F_1\frac{1}{h_1}\frac{\partial(h_2h_3)}{\partial u_1}\frac{1}{h_2h_3} \\
&= \frac{F_1}{h_1h_2h_3}\frac{\partial(h_2h_3)}{\partial u_1}
\end{aligned}$$

The first equality is because the $\hat{\mathbf{u}}_i$ form a right handed system. The second equality holds by equation (3). The third equality is just pulling scalars out of a cross product. The fourth equality use the product rule for the operator $\nabla\cdot$. The fifth equality is because $\nabla \cdot (\nabla f \times \nabla g)$ vanishes (see below), and uses equation (3) to convert the first term back to $\hat{\mathbf{u}}_i$ vectors. The second to last equality uses equation (1) to compute the first component of $\nabla(h_2h_3)$.

[Aside: We see that $\nabla \cdot (\nabla f \times \nabla g)$ vanishes because of a vector cross product identity and the fact that $\nabla \times \nabla = 0$. Specifically,

$$\nabla \cdot (\nabla f \times \nabla g) \;=\; \nabla g \cdot (\nabla \times \nabla f) \,-\, \nabla f \cdot (\nabla \times \nabla g) \;=\; 0]$$

So, in the case $i = 1$, equation (4) becomes

$$\begin{aligned}
\nabla \cdot (F_1\hat{\mathbf{u}}_1) &= (\nabla F_1) \cdot \hat{\mathbf{u}}_1 \,+\, F_1(\nabla \cdot \hat{\mathbf{u}}_1) \\
&= \frac{1}{h_1}\frac{\partial F_1}{\partial u_1} + \frac{F_1}{h_1h_2h_3}\frac{\partial(h_2h_3)}{\partial u_1} \\
&= \frac{1}{h_1h_2h_3}\frac{\partial(h_2h_3F_1)}{\partial u_1}
\end{aligned}$$

We obtain similar expressions in the case $i = 2, 3$. Combining all three gives the following expression for the divergence

$$\nabla \cdot \mathbf{F} \;=\; \frac{1}{h_1 h_2 h_3}\left[\frac{\partial(h_2 h_3 F_1)}{\partial u_1} + \frac{\partial(h_1 h_3 F_2)}{\partial u_2} + \frac{\partial(h_1 h_2 F_3)}{\partial u_3}\right] \tag{6}$$

In **cylindrical coordinates** the divergence of $\mathbf{F} = F_1\hat{\mathbf{r}} + F_2\hat{\theta} + F_3\hat{\mathbf{z}}$ is

$$\nabla \cdot \mathbf{F} \;=\; \frac{1}{r}\left[\frac{\partial(rF_1)}{\partial r} + \frac{\partial(F_2)}{\partial \theta} + \frac{\partial(rF_3)}{\partial z}\right]$$

In **spherical coordinates** the divergence of $\mathbf{F} = F_1\hat{\rho} + F_2\hat{\phi} + F_3\hat{\theta}$ is

$$\nabla \cdot \mathbf{F} \;=\; \frac{1}{\rho^2 \sin\phi}\left[\frac{\partial(\rho^2 \sin\phi F_1)}{\partial \rho} + \frac{\partial(\rho \sin\phi F_2)}{\partial \phi} + \frac{\partial(\rho F_3)}{\partial \theta}\right]$$

9. **The Laplacian in Curvilinear Coordinates.** Combining the results from the two previous sections, we get an expression for the Laplacian ($\Delta = \nabla^2$).

$$\begin{aligned}\Delta f \;=\; \nabla^2 f \;&=\; \nabla \cdot \nabla f \\ &=\; \nabla \cdot \left(\frac{1}{h_1}\frac{\partial f}{\partial u_1}\hat{\mathbf{u}}_1 + \frac{1}{h_2}\frac{\partial f}{\partial u_2}\hat{\mathbf{u}}_2 + \frac{1}{h_3}\frac{\partial f}{\partial u_3}\hat{\mathbf{u}}_3\right) \\ &=\; \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial f}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_1 h_3}{h_2}\frac{\partial f}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial f}{\partial u_3}\right)\right]\end{aligned}$$

In **cylindrical coordinates** the Laplacian is

$$\Delta f \;=\; \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial f}{\partial \theta}\right) + \frac{\partial}{\partial z}\left(r\frac{\partial f}{\partial z}\right)\right]$$

In **spherical coordinates** the Laplacian is

$$\Delta f \;=\; \frac{1}{\rho^2 \sin\phi}\left[\frac{\partial}{\partial \rho}\left(\rho^2 \sin\phi \frac{\partial f}{\partial \rho}\right) + \frac{\partial}{\partial \phi}\left(\sin\phi \frac{\partial f}{\partial \phi}\right) + \frac{\partial}{\partial \theta}\left(\frac{1}{\sin\phi}\frac{\partial f}{\partial \theta}\right)\right]$$

10. **The Curl in Curvilinear Coordinates.** Let $\mathbf{F}$ be a vector field with coordinate functions F_i with respect to the unit vectors $\hat{\mathbf{u}}_i$. That is

$$\mathbf{F} \;=\; F_1\hat{\mathbf{u}}_1 + F_2\hat{\mathbf{u}}_2 + F_3\hat{\mathbf{u}}_3$$

where the F_i are functions of (u_1, u_2, u_3).

We compute the curl $\nabla \times \mathbf{F}$ using properties of the differential operator $\nabla\times$. First $\nabla\times$ satisfies a sum rule, and so it suffices to determine each $\nabla \times (F_i\hat{\mathbf{u}}_i)$ individually. Furthermore, the product rule for $\nabla\times$ gives

$$\nabla \times (F_i\hat{\mathbf{u}}_i) \;=\; (\nabla F_i) \times \hat{\mathbf{u}}_i + F_i(\nabla \times \hat{\mathbf{u}}_i) \tag{7}$$

For concreteness, we'll work out the right side of equation (7) in the case $i = 1$. The cases $i = 2, 3$ are similar.

Using equation (1) we can write out the first term on the right side of equation (7) as

$$\begin{aligned}(\nabla F_1)\times\hat{\mathbf{u}}_1 &= \left[\frac{1}{h_1}\frac{\partial F_1}{\partial u_1}\hat{\mathbf{u}}_1 + \frac{1}{h_2}\frac{\partial F_1}{\partial u_2}\hat{\mathbf{u}}_2 + \frac{1}{h_3}\frac{\partial F_1}{\partial u_3}\hat{\mathbf{u}}_3\right]\times\hat{\mathbf{u}}_1 \\ &= \frac{1}{h_3}\frac{\partial F_1}{\partial u_3}\hat{\mathbf{u}}_2 - \frac{1}{h_2}\frac{\partial F_1}{\partial u_2}\hat{\mathbf{u}}_3\end{aligned}$$

The second equality just uses the fact that the $\hat{\mathbf{u}}_i$ form a right handed system.

We can use the gradient version of the $\hat{\mathbf{u}}_i$ (from equation (3)) to write the second term on the right side of equation (7) as

$$\begin{aligned}F_1(\nabla\times\hat{\mathbf{u}}_1) &= F_1(\nabla\times(h_1\nabla u_1)) \\ &= F_1\nabla(h_1)\times\nabla(u_1) + F_1h_1(\nabla\times\nabla u_1) \\ &= F_1\left[\frac{1}{h_1}\frac{\partial h_1}{\partial u_1}\hat{\mathbf{u}}_1 + \frac{1}{h_2}\frac{\partial h_1}{\partial u_2}\hat{\mathbf{u}}_2 + \frac{1}{h_3}\frac{\partial h_1}{\partial u_3}\hat{\mathbf{u}}_3\right]\times\nabla u_1 + \mathbf{0} \\ &= F_1\left[\frac{1}{h_1}\frac{\partial h_1}{\partial u_1}\hat{\mathbf{u}}_1 + \frac{1}{h_2}\frac{\partial h_1}{\partial u_2}\hat{\mathbf{u}}_2 + \frac{1}{h_3}\frac{\partial h_1}{\partial u_3}\hat{\mathbf{u}}_3\right]\times\frac{\hat{\mathbf{u}}_1}{h_1} \\ &= \frac{F_1}{h_1h_3}\frac{\partial h_1}{\partial u_3}\hat{\mathbf{u}}_2 - \frac{F_1}{h_1h_2}\frac{\partial h_1}{\partial u_2}\hat{\mathbf{u}}_3\end{aligned}$$

Combining the results of the past two paragraphs we get that equation (7) becomes

$$\begin{aligned}\nabla\times(F_1\hat{\mathbf{u}}_1) &= \left(\frac{1}{h_3}\frac{\partial F_1}{\partial u_3} + \frac{F_1}{h_1h_3}\frac{\partial h_1}{\partial u_3}\right)\hat{\mathbf{u}}_2 - \left(\frac{1}{h_2}\frac{\partial F_1}{\partial u_2} + \frac{F_1}{h_1h_2}\frac{\partial h_1}{\partial u_2}\right)\hat{\mathbf{u}}_3 \\ &= \left(\frac{1}{h_1h_3}\frac{\partial(h_1F_1)}{\partial u_3}\right)\hat{\mathbf{u}}_2 - \left(\frac{1}{h_1h_2}\frac{\partial(h_1F_1)}{\partial u_2}\right)\hat{\mathbf{u}}_3\end{aligned}$$

There are similar expressions for the case $i = 2, 3$. We recognize sum of all these as the output of a 3×3–determinant, and so obtain the the following expression for the curl

$$\nabla\times\mathbf{F} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{\mathbf{u}}_1 & h_2\hat{\mathbf{u}}_2 & h_3\hat{\mathbf{u}}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1F_1 & h_2F_2 & h_3F_3\end{vmatrix} \tag{8}$$

In **cylindrical coordinates** the Curl of $\mathbf{F} = F_1\hat{\mathbf{r}} + F_2\hat{\theta} + F_3\hat{\mathbf{z}}$ is

$$\nabla\times\mathbf{F} = \frac{1}{r}\begin{vmatrix}\hat{\mathbf{r}} & r\hat{\theta} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z} \\ F_1 & rF_2 & F_3\end{vmatrix}$$

In **spherical coordinates** the Curl of $\mathbf{F} = F_1\hat{\rho} + F_2\hat{\phi} + F_3\hat{\theta}$ is

$$\nabla\times\mathbf{F} = \frac{1}{\rho^2\sin\phi}\begin{vmatrix}\hat{\rho} & \rho\hat{\phi} & \rho\sin\phi\hat{\theta} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial\theta} \\ F_1 & \rho F_2 & \rho\sin\phi F_3\end{vmatrix}$$