Q28 (p1062)

Average distance from point in ball of radius a to its center.

distance to center $= \rho$

Ball is $0 \le \rho \le a \leftarrow B$

Want average of $f(\rho, \phi, \theta) = \rho$ over the ball B.

$$\hat{f} = \frac{\iiint_B f \, dV}{\iiint_B dV} = \frac{\iiint_B \rho \, dV}{\frac{4}{3}\pi a^3}$$

Geometry (or previous calculation)

Now...

$$\iiint_B \rho \, dV = \int_0^{2\pi} \int_0^{\pi} \int_0^{a} \rho \, (\rho^2 \sin\phi) \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\pi} \sin\phi \, d\phi \int_0^{a} \rho^3 \, d\rho$$

$$= (2\pi) \cdot \left[-\cos\phi\right]_0^{\pi} \cdot \left(\frac{a^4}{4}\right)$$

$$= 2\pi(2)\frac{a^4}{4} = \pi a^4$$

Therefore, ...

$$\hat{f} = \frac{\cancel{\pi} a^{\cancel{4}} \cdot 3}{4 \cancel{\pi} \cancel{a^3}} = \boxed{\frac{3a}{4}}$$

SPHERICAL

Q41 (pge 1062)

$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{2-\sqrt{4-x^2-y^2}}^{2+\sqrt{4-x^2-y^2}} (x^2+y^2+z^2)^{3/2}\, dz\, dy\, dx$$

Disk

$x^2+y^2 \le 4$

center $(0,0)$

radius 2

$$z = 2 \pm \sqrt{4-x^2-y^2}$$

$$(z-2) = \pm\sqrt{4-x^2y^2}$$

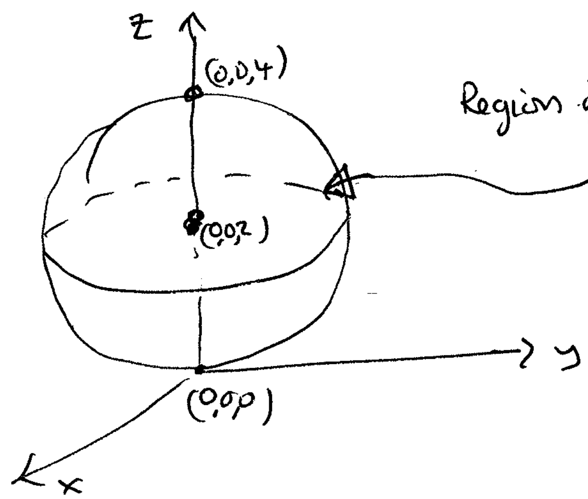
$$x^2+y^2+(z-2)^2 = 4$$

Sphere

Center $(0,0,2)$

Radius 2

SPHERICAL



Region of integration, E

Polar description of the sphere: $\rho^2\sin^2\phi + (\rho\cos\phi - 2)^2 = 4$

$$\rho^2(\sin^2\phi + \cos^2\phi) + 2(-2)(\rho\cos\phi) + \cancel{(-2)^2} = \cancel{4}$$

$$\rho^2 - 4\rho\cos\phi = 0$$

$$\rho\,(\underline{\rho - 4\cos\phi}) = 0$$

$$\boxed{0 \le \rho \le 4\cos\phi}$$

$$\boxed{0 \le \phi \le \pi/2}$$

$$\boxed{0 \le \theta \le 2\pi}$$

Spherical polar description of the spherical Region, E.

No part of region is below xy-plane.

Integrand $(x^2+y^2+z^2)^{3/2} = (\rho^2)^{3/2} = \rho^3$

So integral becomes...

$$\int_0^{2\pi}\int_0^{\pi/2}\int_0^{4\cos\phi} \rho^3 \quad \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\pi/2} \left[\frac{\rho^6}{6}\right]_0^{4\cos\phi} \sin\phi \, d\phi$$

$$= \frac{2\pi\,(4)^6}{6} \int_0^{\pi/2} \cos^6\phi \, \sin\phi \, d\phi$$

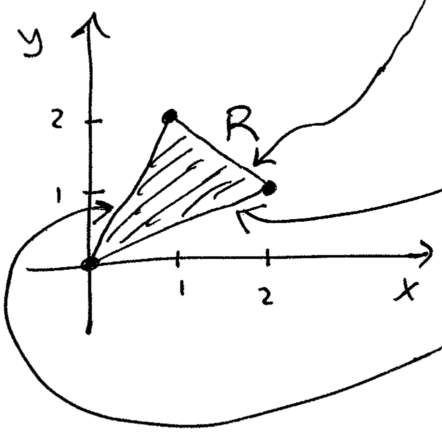
$$= \frac{\pi}{3}\,(4)^6 \left[-\frac{\cos^7\phi}{7}\right]_0^{\pi/2}$$

$$= \frac{(4)^6\,\pi}{21}$$

SPHERICAL

Q15 (p. 1071)

GENERAL COORDS



Line $x+y=3 \rightsquigarrow (2u+v)+(u+2v)=3$

$\Rightarrow \boxed{u+v=1}$

Line $y=\frac{x}{2} \rightsquigarrow (u+2v)=\frac{1}{2}(2u+v)$

$\Rightarrow 2(u+2v)=2u+v$

$\Rightarrow \cancel{2u}+4v=\cancel{2u}+v$

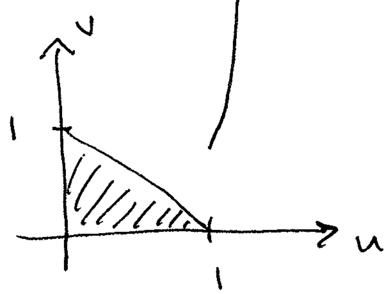
$\Rightarrow 3v=0$ $\boxed{v=0}$

Line $y=2x \rightsquigarrow (u+2v)=2(2u+v)$

$u+\cancel{2v}=4u+\cancel{2v}$

$3u=0 \Rightarrow \boxed{u=0}$

$\Phi(u,v)=(2u+v,\ u+2v)$



Now $\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}\right|$

$= \left|\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}\right| = |(2)(2)-(1)(1)| = \boxed{3}$

Therefore,----

$$\iint_R (x-3y)\,dA = \iint [(2u+v)-3(u+2v)] \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

$$= \int_0^1 \left(\int_0^{1-v} (-u-5v)\, 3\, du\right) dv$$

$$= -3 \int_0^1 \left[\frac{u^2}{2}+5vu\right]_0^{1-v} dv$$

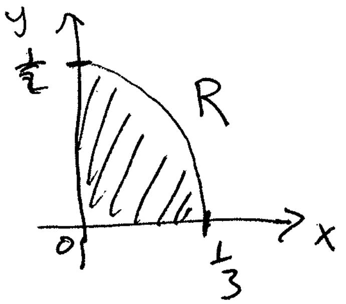
$$= -\frac{3}{2}\int_0^1 (1-v)^2+10v(1-v)\,dv = \cdots = \boxed{-3}$$

Q2b (p. 1072)

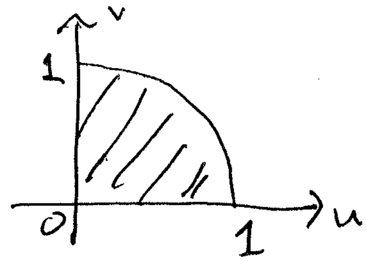
$u = 3x \qquad v = 2y \qquad \boxed{x = u/3 \qquad y = v/2}$

GENERAL COORDS

$9x^2 + 4y^2 = 1 \rightsquigarrow u^2 + v^2 = 1$



$\Phi(u,v) = \left(\frac{u}{3}, \frac{v}{2}\right)$



$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}\right| = \left|\begin{vmatrix} 1/3 & 0 \\ 0 & \frac{1}{2} \end{vmatrix}\right| = \left|\left(\tfrac{1}{3}\right)\left(\tfrac{1}{2}\right) - 0^2\right| = \boxed{\frac{1}{6}}$$

$$\iint_R \sin(9x^2 + 4y^2)\, dA = \iint \sin(u^2 + v^2) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv$$

$$= \frac{1}{6} \iint \sin(u^2+v^2)\, du\, dv$$

Now ...
Switch to Polar Coords!

$r^2 = u^2 + v^2$

$\boxed{\begin{array}{l} u = r\cos\theta \\ v = r\sin\theta \end{array}}$

$\boxed{\begin{array}{l} 0 \le \theta \le \pi/2 \\ 0 \le r \le 1 \end{array}}$

$$= \frac{1}{6} \int_0^{\pi/2} \int_0^1 \sin(r^2)\, r\, dr\, d\theta$$

$$= \frac{1}{6} \left[\theta\right]_0^{\pi/2} \left[-\frac{\cos(r^2)}{2}\right]_0^1$$

$$= \boxed{\frac{\pi}{24}\left(1 - \cos(1)\right)}$$