Clay / Whiteway

Burnside's lemma

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Burnside's lemma, sometimes also called **Burnside's counting theorem**, the **Cauchy-Frobenius lemma** or the **orbit-counting theorem**, is a result in group theory which is often useful in taking account of symmetry when counting mathematical objects. Its various eponyms include William Burnside, George Pólya, Augustin Louis Cauchy, and Ferdinand Georg Frobenius. The result is not due to Burnside himself, who merely quotes it in his book 'On the Theory of Groups of Finite Order', attributing it instead to Frobenius (1887).[1]

In the following, let G be a finite group that acts on a set X. For each g in G let X^g denote the set of elements in X that are fixed by g. Burnside's lemma asserts the following formula for the number of orbits, denoted $|X/G|$:[2]

$$|X/G| = \frac{1}{|G|}\sum_{g\in G}|X^g|.$$

Thus the number of orbits (a natural number or $+\infty$) is equal to the average number of points fixed by an element of G (which consequently is also a natural number or infinity).

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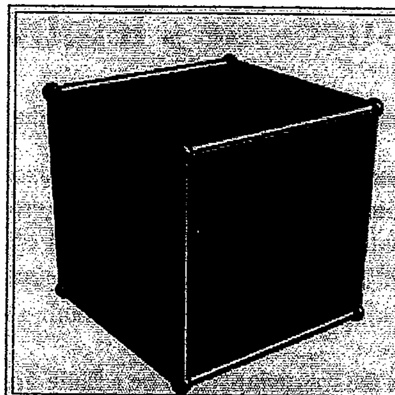
Example application

The number of rotationally distinct colourings of the faces of a cube using three colours can be determined from this formula as follows.

Let X be the set of 3^6 possible face colour combinations that can be applied to a cube in one particular orientation, and let the rotation group G of the cube act on X in the natural manner. Then two elements of X belong to the same orbit precisely when one is simply a rotation of the other. The number of rotationally distinct colourings is thus the same as the number of orbits and can be found by counting the sizes of the fixed sets for the 24 elements of G.

- one identity element which leaves all 3^6 elements of X unchanged
- six 90-degree face rotations, each of which leaves 3^3 of the elements of X unchanged
- three 180-degree face rotations, each of which leaves 3^4 of the elements of X unchanged
- eight 120-degree vertex rotations, each of which leaves 3^2 of the elements of X unchanged
- six 180-degree edge rotations, each of which leaves 3^3 of the elements of X unchanged


Cube with coloured faces

A detailed examination of these automorphisms may be found here.

The average fix size is thus

$$\frac{1}{24}\left(3^6+6\times 3^3+3\times 3^4+8\times 3^2+6\times 3^3\right)=57.$$

Hence there are 57 rotationally distinct colourings of the faces of a cube in three colours. In general, the number of rotationally distinct colorings of the faces of a cube in n colors is given by

$$\frac{1}{24}\left(n^6+3n^4+12n^3+8n^2\right)$$

Proof

The proof uses the orbit-stabilizer theorem and the fact that X is the disjoint union of the orbits:

$$\begin{aligned}\sum_{g\in G}|X^g| &= |\{(g,x)\in G\times X \mid g\cdot x = x\}| = \sum_{x\in X}|G_x| = \sum_{x\in X}\frac{|G|}{|Gx|}\\ &= |G|\sum_{x\in X}\frac{1}{|Gx|} = |G|\sum_{A\in X/G}\sum_{x\in A}\frac{1}{|A|} = |G|\sum_{A\in X/G}1\\ &= |G|\cdot|X/G|.\end{aligned}$$

History: the lemma that is not Burnside's

William Burnside stated and proved this lemma, attributing it to Frobenius 1887 in his 1897 book on finite groups. But even prior to Frobenius, the formula was known to Cauchy in 1845. In fact, the lemma was apparently so well known that Burnside simply omitted to attribute it to Cauchy. Consequently, this lemma is sometimes referred to as the **lemma that is not Burnside's**.[3] This is less ambiguous than it may seem: Burnside contributed many lemmas to this field.

Notes

1. ^ Burnside 1897, §119
2. ^ Rotman 1995, Chapter 3
3. ^ Neumann 1979

See also

- Pólya enumeration theorem

References

- Burnside, William (1897), *Theory of groups of finite order*, Cambridge University Press. (This is the first edition; the introduction to the second edition contains Burnside's famous *volte face* regarding the utility of representation theory.)
- Frobenius, Ferdinand Georg (1887), "Ueber die Congruenz nach einem aus zwei endlichen Gruppen gebildeten Doppelmodul", *Crelle* **CI**: 288.
- Neumann, Peter M. (1979), "A lemma that is not Burnside's", *The Mathematical Scientist* **4** (2): 133–141, MR562002 (http://www.ams.org/mathscinet-getitem?mr=562002) , ISSN 0312-3685 (http://www.worldcat.org/issn/0312-3685) .
- Rotman, Joseph (1995), *An introduction to the theory of groups*, Springer-Verlag, ISBN 0-387-94285-8.