Q1]... [10 points] Sketch the level curves $f = 1$, $f = 4$, $f = 0$, $f = -1$, and $f = -4$ of the function $f(x, y) = x^2 - 4y^2$.

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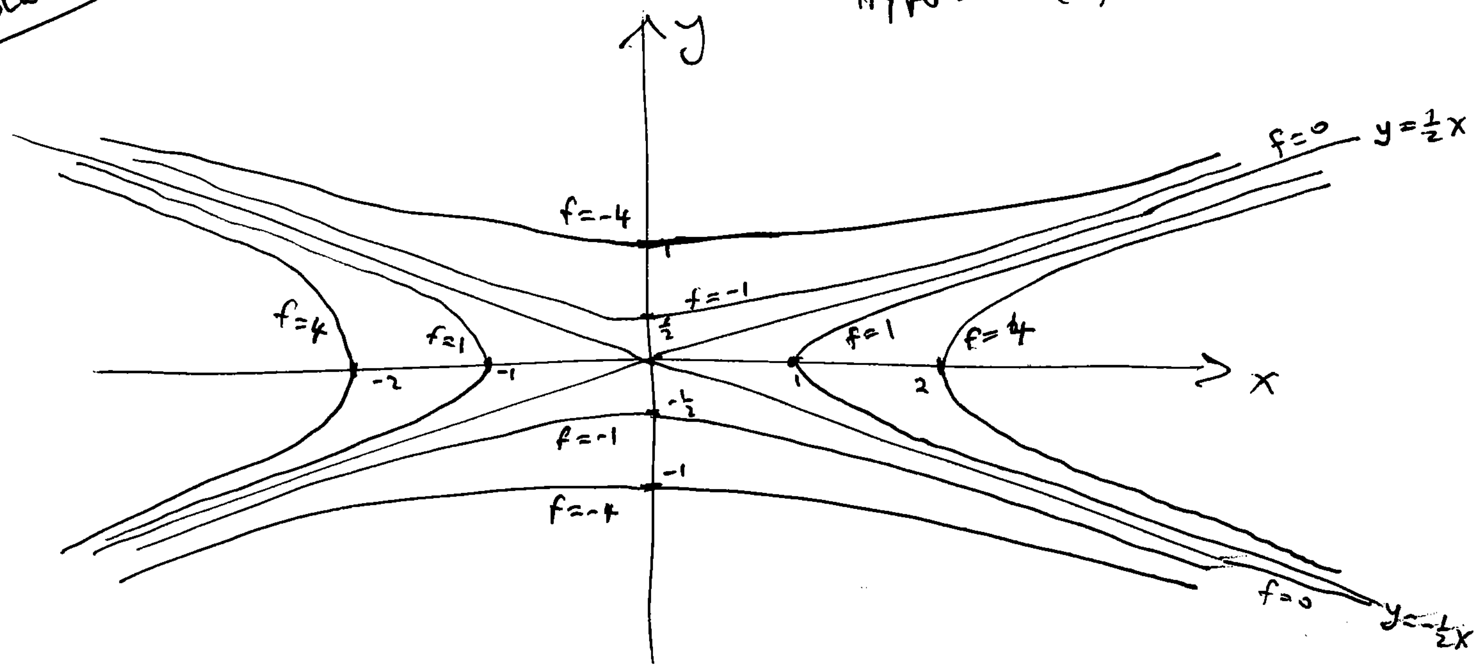
Hyperbolas (asymptotic to $y = \pm \frac{1}{2}x$)



Q2]... [10 points] Does the following limit exist? Give reasons for your answer.

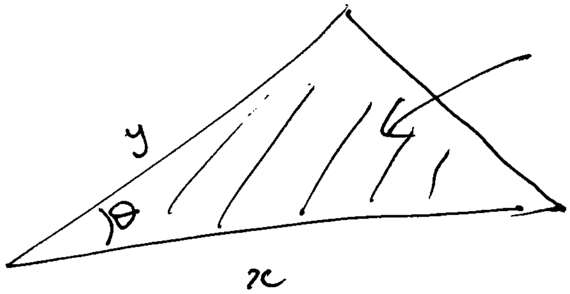
$$\lim_{(x,y)\to(0,0)} \frac{4xy}{x^2 + 2y^2}$$

$\lim\limits_{\substack{(x,y)\to(0,0)\\ \text{along } x\text{-axis } (y=0)}} \dfrac{4xy}{x^2+2y^2} = \lim\limits_{x\to 0} \dfrac{0}{x^2} = \lim\limits_{x\to 0} (0) = 0$

$\lim\limits_{\substack{(x,y)\to(0,0)\\ \text{along } y=x}} \dfrac{4xy}{x^2+2y^2} = \lim\limits_{x\to 0} \dfrac{4x^2}{3x^2} = \lim\limits_{x\to 0} \left(\dfrac{4}{3}\right) = \dfrac{4}{3}$

These don't agree $\Rightarrow$ $\boxed{\lim\limits_{(x,y)\to(0,0)} \text{D.N.E.}}$

Q3]... [12 points] The area of a triangular field of side lengths x, y and contained angle θ is given by $A = \frac{1}{2}xy\sin\theta$. Write down the differential dA, and use it to estimate the error in the area of a field with side measurements of 150m and 200m (each accurate to within ±1m) and contained angle of 30° (accurate to within $\pm2°$).



$$\text{Area} = \frac{1}{2}xy\sin\theta \quad \begin{cases} \dfrac{\partial A}{\partial x} = \dfrac{y}{2}\sin\theta \\ \dfrac{\partial A}{\partial y} = \dfrac{x}{2}\sin\theta \\ \dfrac{\partial A}{\partial \theta} = \dfrac{xy}{2}\cos\theta \end{cases}$$

$$dA = \frac{\partial A}{\partial x}dx + \frac{\partial A}{\partial y}dy + \frac{\partial A}{\partial \theta}d\theta$$

$$= \frac{1}{2}\left[y\sin\theta\, dx + x\sin\theta\, dy + xy\cos\theta\, d\theta\right]$$

$$= \frac{1}{2}\left[(200)\left(\frac{1}{2}\right)(1) + (150)\left(\frac{1}{2}\right)(1) + (200)(150)\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\pi}{90}\right)\right]$$

$x = 150$
$y = 200$
$dx = dy = 1$
$\theta = \frac{30}{180}\cdot\pi = \pi/6$
$d\theta = \frac{2}{180}\cdot\pi = \frac{\pi}{90}$

$$= \frac{1}{2}\left[100 + 75 + \frac{500\pi}{\sqrt{3}}\right]$$

$$= \frac{1}{2}\left(175 + \frac{500\pi}{\sqrt{3}}\right) m^2.$$

$\sin(\pi/6) = \frac{1}{2}$
$\cos(\pi/6) = \frac{\sqrt{3}}{2}$

Q4]... [**12 points**] Write down ∇f for the function $f(x, y, z) = x^2y^3z^4$.

$$\nabla f = \langle f_x, f_y, f_z \rangle$$
$$= \langle 2xy^3z^4, \; 3x^2y^2z^4, \; 4x^2y^3z^3 \rangle$$

In what direction is f increasing most rapidly at the point (2,1,3)?

In the $\nabla f(2,1,3) = \langle (18)^2, 4(3^5), 16(3)^3 \rangle$

What is the value of the maximum rate of change of f at the point $(2, 1, 3)$?

$$\text{Ans} = |\nabla f| = \left(\sqrt{(2(1)(3))^2 + (3(2)(3))^2 + (4(2)(1))^2}\right)(2)(1)^2(3)^3$$

$$= (4)(27)\sqrt{9 + 81 + 16}$$

$$= (4)(27)\sqrt{106}$$

Q5]... [**13 points**] Prove that the two surfaces $x^2 + y^2 + z^2 = 25$ and $x^2 = 4y^2 + 4z^2$ are perpendicular (orthogonal) to each other at all points of intersection.

$$F(x,y,z) = x^2 + y^2 + z^2$$

1^{st} surface is the <u>Level surface</u> $\boxed{F = 25}$

$$G(x,y,z) = -x^2 + 4y^2 + 4z^2$$

2^{nd} surface is the <u>Level surface</u> $\boxed{G = 0}$

Normals to surfaces are ∇F, ∇G.

$$\nabla F = \langle 2x, 2y, 2z \rangle$$

$$\nabla G = \langle -2x, 8y, 8z \rangle$$

$$\nabla F \cdot \nabla G = -4x^2 + 16y^2 + 16z^2$$

$$= 4(-x^2 + 4y^2 + 4z^2)$$

$$= 4(0) = 0$$

since point belongs to <u>$G(x,y,z) = 0$</u>.

$\Rightarrow$ Normals $\perp$ to each other.

$\Rightarrow$ surfaces intersect orthogonally.

Q6]... **[18 points]** Suppose that $z = f(x,y)$ where $x = g(s,t)$ and $y = h(s,t)$. Verify that

$$\frac{\partial^2 z}{\partial t^2} = \frac{\partial^2 z}{\partial x^2}\left(\frac{\partial x}{\partial t}\right)^2 + 2\frac{\partial^2 z}{\partial x \partial y}\frac{\partial x}{\partial t}\frac{\partial y}{\partial t} + \frac{\partial^2 z}{\partial y^2}\left(\frac{\partial y}{\partial t}\right)^2 + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial t^2} + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial t^2}$$

Show all the steps involved. Also, write down (no work needs to be shown) a similar expression for $\frac{\partial^2 z}{\partial s^2}$.

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$\frac{\partial^2 z}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}\right)$$

$$= \frac{\partial}{\partial t}\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial x}{\partial t}\right) + \frac{\partial z}{\partial x}\frac{\partial}{\partial t}\left(\frac{\partial x}{\partial t}\right) + \frac{\partial}{\partial t}\left(\frac{\partial z}{\partial y}\right)\left(\frac{\partial y}{\partial t}\right) + \frac{\partial z}{\partial y}\frac{\partial}{\partial t}\left(\frac{\partial y}{\partial t}\right)$$

Apply Chain Rule to ~ terms.

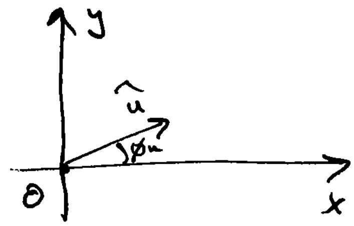
$$= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)\frac{\partial x}{\partial t}\left(\frac{\partial x}{\partial t}\right) + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)\frac{\partial y}{\partial t}\left(\frac{\partial x}{\partial t}\right) + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial t^2} + \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)\frac{\partial x}{\partial t}\left(\frac{\partial y}{\partial t}\right) + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\frac{\partial y}{\partial t}\left(\frac{\partial y}{\partial t}\right) + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial t^2}$$

$$= \frac{\partial^2 z}{\partial x^2}\left(\frac{\partial x}{\partial t}\right)^2 + \frac{\partial^2 z}{\partial y^2}\left(\frac{\partial y}{\partial t}\right)^2 + 2\frac{\partial^2 z}{\partial x \partial y}\left(\frac{\partial x}{\partial t}\right)\left(\frac{\partial y}{\partial t}\right) + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial t^2} + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial t^2}$$

$$\frac{\partial^2 z}{\partial s^2} = \frac{\partial^2 z}{\partial x^2}\left(\frac{\partial x}{\partial s}\right)^2 + \frac{\partial^2 z}{\partial y^2}\left(\frac{\partial y}{\partial s}\right)^2 + 2\frac{\partial^2 z}{\partial x \partial y}\left(\frac{\partial x}{\partial s}\right)\left(\frac{\partial y}{\partial s}\right) + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial s^2} + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial s^2}$$

Bonus]... Suppose that $f(x,y)$ is differentiable at the point (a,b), and that you are told the values of the directional derivatives, $D_{\mathbf{u}}f(a,b)$ and $D_{\mathbf{v}}f(a,b)$, of f at the point (a,b) in the directions specified by the unit vectors $\mathbf{u}$ and $\mathbf{v}$. Suppose that $\mathbf{u}$ (resp. $\mathbf{v}$) makes an angle ϕ_u (resp. ϕ_v) with the positive x-axis. What (minimal) condition must ϕ_u and ϕ_v satisfy in order to reclaim the values of $f_x(a,b)$ and $f_y(a,b)$? Show how to compute the values of $f_x(a,b)$ and $f_y(a,b)$ from the two directional derivatives above.

Let $\boxed{\nabla f = \langle P, Q \rangle}$



$$\hat{u} = \langle \cos(\phi_u), \sin(\phi_u) \rangle$$

$$\hat{v} = \langle \cos(\phi_v), \sin(\phi_v) \rangle$$

$$D_u f(a,b) = \nabla f(a,b) \cdot \hat{u}$$

$$= \langle P, Q \rangle \cdot \langle \cos(\phi_u), \sin(\phi_u) \rangle$$

$$= P \cos\phi_u + Q \sin\phi_u$$

$$D_v f(a,b) = \nabla f(a,b) \cdot \hat{v}$$

$$= (P, Q) \cdot (\cos\phi_v, \sin\phi_v)$$

$$= P\cos(\phi_v) + Q \sin(\phi_v)$$

$$(*) \text{---} \begin{cases} P\cos\phi_u + Q\sin\phi_u = D_u f(a,b) & \leftarrow \text{times } \sin\phi_v \\ P\cos\phi_v + Q\sin\phi_v = D_v f(a,b) & \leftarrow \text{times } \sin\phi_u \end{cases}$$

subtract.

$$P(\cos\phi_u \sin\phi_v - \cos\phi_v \sin\phi_u) + 0 = D_u f(a,b) \sin\phi_v - D_v f(a,b) \sin\phi_u$$

$$P \sin(\phi_u - \phi_v) = \text{RHS}...$$

$$P = \frac{\text{RHS}}{\sin(\phi_u - \phi_v)}$$

Need $\sin(\phi_u - \phi_v) \neq 0$

ie $\phi_u \neq \phi_v$

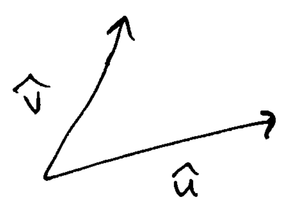
$\phi_u \neq \phi_v + \pi$

That is $\hat{u} \nparallel \hat{v}$ and $\hat{u} \nparallel -\hat{v}$

vectors not parallel or pointing in opposite directions.

There's a similar procedure for eliminating P from equations $(*)$. $(\text{Top})\cos\phi_v - (\text{Lower})(\cos\phi_u)$

Again, the result for Q will involve $\div$ by $\underline{\sin(\phi_u - \phi_v)}$.



<u>Summary:</u> Provided f is differentiable, then any pair of directional (not parallel/antiparallel) derivatives will be sufficient to reclaim $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$ values.
