Q1]... **[10 points]** Is the function

$$f(x,y) = \begin{cases} \frac{x^3 y}{x^6+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

continuous at the point $(0,0)$? Give reasons for your answer.

[S09, MT1 — SOLNS]

No.

$$\lim_{\substack{(x,y)\to(0,0) \\ \text{along } x\text{-axis}}} f(x,y) = \lim_{x\to 0} \frac{0}{x^6} = \lim_{x\to 0} (0) = 0$$

$$\lim_{\substack{(x,y)\to(0,0) \\ \text{along } y=x^3}} f(x,y) = \lim_{x\to 0} \frac{x^6}{x^6+x^6} = \lim_{x\to 0} \left(\frac{1}{2}\right) = \frac{1}{2}$$

These don't agree $\Rightarrow \lim_{(x,y)\to(0,0)} f(x,y)$ D.N.E.

$\Rightarrow f(x,y)$ not continuous at $(0,0)$.

Q2]... **[30 points]** Let $f(x, y, z) = x^2 + 2y^2 + 3z^2$.

(a) In what direction is f increasing most rapidly at the point $(1, 1, 1)$, and what is this rate of increase?

Increasing most rapidly in $\nabla f_{(1,1,1)}$ direction

$$= \langle 2x, 4y, 6z \rangle_{(1,1,1)}$$

$$= \langle 2, 4, 6 \rangle$$

Rate of increase $= |\nabla f_{(1,1,1)}| = |\langle 2, 4, 6 \rangle|$

$$= 2\sqrt{14}$$

(b) Find the equation of the tangent plane to the surface $f(x, y, z) = 6$ at the point $(1, 1, 1)$.

Normal $= \nabla f_{(1,1,1)}$ Point $= (1, 1, 1)$

Eqn $\Rightarrow$ $\nabla f_{(1,1,1)} \cdot \langle x-1, y-1, z-1 \rangle = 0$

$$\boxed{1(x-1) + 2(y-1) + 3(z-1) = 0}$$

(c) What is the rate of change of f at $(1, 1, 1)$ in the direction from $(1, 1, 1)$ to $(-1, 0, 2)$?

$$\hat{u} = \frac{\langle -1, 0, 2 \rangle - \langle 1, 1, 1 \rangle}{|\langle -1, 0, 2 \rangle - \langle 1, 1, 1 \rangle|} = \frac{\langle -2, -1, 1 \rangle}{\sqrt{6}}$$

Rate of change $= D_{\hat{u}} f_{(1,1,1)}$

$$= \nabla f_{(1,1,1)} \cdot \hat{u}$$

$$= \langle 2, 4, 6 \rangle \cdot \frac{\langle -2, -1, 1 \rangle}{\sqrt{6}}$$

$$= \boxed{\frac{-2}{\sqrt{6}}}$$

Q3]...[20 points] If $f(u, v, w)$ is differentiable, and $u = x - y$, $v = y - z$, $w = z - x$, then show that

$$f_x + f_y + f_z = 0\,.$$

Show all the steps of your work clearly.

$$f_x = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x} \qquad \text{Ch. Rule}$$

$$= f_u(1) + f_v(0) + f_w(-1)$$

$$f_y = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial y} \qquad \text{Ch. Rule}$$

$$= f_u(-1) + f_v(1) + f_w(0)$$

$$f_z = \frac{\partial f}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial z} \qquad \text{Ch. Rule}$$

$$= f_u(0) + f_v(-1) + f_w(1)$$

Adding ....

$$f_x + f_y + f_z = (f_u) + (-f_w) + (-f_u) + (f_v)$$

$$+ (-f_v) + (f_w)$$

$$= 0$$

Q4]... **[25 points]** State the second derivative test for the function $f(x, y)$ at the critical point (a, b).

$$D = f_{xx} f_{yy} - (f_{xy})^2$$

$D(a,b) > 0 \quad f_{xx}(a,b) > 0 \quad \Rightarrow$ LOCAL MIN

$D(a,b) > 0 \quad f_{xx}(a,b) < 0 \quad \Rightarrow$ Local Max

$D(a,b) < 0 \quad \Rightarrow$ SADDLE POINT

$D(a,b) = 0 \quad \Rightarrow$ NO CONCLUSION

The function $f(x, y) = 2x^3 - 6xy + 3y^2$ has two critical points. Find these critical points, and then use the second derivative test to classify them.

$f_x = 6x^2 - 6y \qquad f_y = -6x + 6y$

$f_y = 0 \Rightarrow x = y$

$f_x = 0$

$\Rightarrow x^2 - x = 0$

$x(x-1) = 0$

$x = 0, (y=0), \quad x = 1 \ (y=1) \quad \Rightarrow (0,0), (1,1)$

$f_{xx} = 12x \qquad f_{yy} = 6 \qquad f_{xy} = -6$

$D = 72x - (-6)^2 = 72x - 36$

$D(0,0) = -36 < 0 \quad \Rightarrow$ SADDLE at $(0,0)$.

$D(1,1) = 72 - 36 = 36 > 0, \quad f_{xx}(1,1) = 12 > 0$

$\Rightarrow$ LOCAL MIN at $(1,1)$.

Q5]... [**15 points**] Suppose that the function $f(x,y)$ is differentiable at all points of the plane. Suppose that $D_{\mathbf{u}}f(0,0) = 3\sqrt{2}$ where $\mathbf{u}$ points in the direction from $(0,0)$ to $(1,1)$, and that $D_{\mathbf{v}}f(0,0) = -3\sqrt{5}$ where $\mathbf{v}$ points in the direction from $(0,0)$ to $(2,-1)$.

Find the values of the partial derivatives $f_x(0,0)$ and $f_y(0,0)$.

$$\nabla f = \langle f_x, f_y \rangle$$

write f_x for $f_x(0,0)$ and f_y for $f_y(0,0)$

$$D_{\hat{u}} f(0,0) = 3\sqrt{2} \Rightarrow \langle f_x, f_y \rangle \cdot \left\langle \tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}} \right\rangle = 3\sqrt{2}$$

$$\boxed{f_x + f_y = 6} \quad \text{— (i)}$$

$$D_{\hat{v}} f(0,0) = -3\sqrt{5} \Rightarrow \langle f_x, f_y \rangle \cdot \left\langle \tfrac{2}{\sqrt{5}}, \tfrac{-1}{\sqrt{5}} \right\rangle = -3\sqrt{5}$$

$$\boxed{2f_x - f_y = -15} \quad \text{— (ii)}$$

Add (i) & (ii) $\Rightarrow$

$$3f_x = -9 \qquad \boxed{f_x = -3} \underset{(i)}{\Rightarrow} \boxed{f_y = 9}$$

What is the rate of change (with respect to time) of f at $(0,0)$ as measured by an observer who moves through $(0,0)$ with velocity $\langle 2,1 \rangle$?

$$\begin{aligned} \text{Rate of change} &= \nabla f(0,0) \cdot \langle 2,1 \rangle \\ &= \langle -3, 9 \rangle \cdot \langle 2,1 \rangle \\ &= -6 + 9 \\ &= 3 \end{aligned}$$