SP '08 — CALC II — MIDTERM II

Q1]... [7 points] Use logarithmic differentiation to compute the derivative y' of the following function

$$y = \sqrt{x}^{\sqrt{x}}$$

$$\ln(y) = \ln\left(\sqrt{x}^{\sqrt{x}}\right) = \sqrt{x}\,\ln\left(\sqrt{x}\right)$$

$$\frac{1}{y}y' = \frac{1}{2\sqrt{x}}\ln\left(\sqrt{x}\right) + \cancel{\sqrt{x}}\,\frac{1}{\cancel{\sqrt{x}}}\,\frac{1}{2\sqrt{x}}$$

$$= \frac{\ln(\sqrt{x}) + 1}{2\sqrt{x}}$$

$$y' = \sqrt{x}^{\sqrt{x}}\left(\frac{\ln(\sqrt{x}) + 1}{2\sqrt{x}}\right)$$

Q2]... [8 points] Two twice differentiable functions f and g are inverses of each other. You are told that $f(4) = 7$, $f'(4) = 3$ and $f''(4) = 2$. Find the values of $g'(7)$ and of $g''(7)$.

$$\boxed{g'(x) = \frac{1}{f'(g(x))}} \xrightarrow{\text{let } x=7} \boxed{g'(7) = \frac{1}{f'(4)} = \frac{1}{3}}$$

$$\Big\downarrow \frac{d}{dx}$$

$$g''(x) = -1\left(f'(g(x))\right)^{-2} f''(g(x))\,g'(x)$$

$$= -\frac{f''(g(x))}{\left[f'(g(x))\right]^3}$$

Now, let $x = 7$

$$\boxed{g''(7) = -\frac{f''(4)}{\left[f'(4)\right]^3} = \frac{-2}{27}}$$

Q3]... [**7 points**] P dollars is invested in a bank account that earns interest at a fixed rate, compounded continuously. Suppose that it takes 7 years for P to double in value (become $2P$). How long would it take for P to triple in value (become $3P$)?

$$\text{Yield} = Pe^{rt}$$

$$2\cancel{P} = \cancel{P}e^{r(7)} \qquad \ln(2) = r\,7$$

$$\boxed{r = \frac{\ln(2)}{7}}$$

Find T so that

$$3\cancel{P} = \cancel{P}e^{rT}$$

$$\ln(3) = rT = \frac{\ln(2)}{7}T$$

$$\Rightarrow \boxed{T = \frac{7\ln(3)}{\ln(2)}} \text{ years.}$$

Q4]... [**8 points**] Compute the derivative y' of the following function

$$y = x\tanh^{-1}(x) + \frac{1}{2}\ln(1-x^2)$$

$$y' = 1.\tanh^{-1}(x) + x.\frac{1}{1-x^2} + \frac{1}{2}\frac{1}{(1-x^2)}(-2x)$$

$$= \tanh^{-1}(x) + \cancel{\frac{x}{1-x^2}} - \cancel{\frac{x}{1-x^2}}$$

$$\boxed{y' = \tanh^{-1}(x)}$$

Q5]...[12 points] The function

$$y = \frac{\ln(x)}{x}$$

has a unique local maximum (which is also its absolute maximum).

(a) Find the value of x where this maximum occurs.

$\rightsquigarrow y' = 0 \quad \Rightarrow \frac{1}{x} \cdot \frac{1}{x} + \ln(x)\left(-\frac{1}{x^2}\right) = 0$

$\Rightarrow \frac{1 - \ln(x)}{x^2} = 0$

$\Rightarrow \ln(x) = 1 \qquad x = e^1 = e$

At $\boxed{x = e}$

(b) Say (giving a reason) which is bigger, $\frac{\ln(e)}{e}$ or $\frac{\ln(\pi)}{\pi}$.

local (absolute) max at e

$\Rightarrow \frac{\ln(e)}{e} > \frac{\ln(\pi)}{\pi}$

(c) Say (giving a reason) which is bigger, e^π or π^e.

Multiply previous by πe (positive number)

$\pi \ln(e) > e \ln(\pi)$

$\ln(e^\pi) > \ln(\pi^e)$ ... Rule of logs

$e^\pi > \pi^e$ ... $\ln(x)$ is increasing

Q6]... [**14 points**] Find the derivative of the function $y = \sin^{-1}(x)$. Show your work.

$$y = \sin^{-1}(x)$$

$$x = \sin(y)$$

$$1 = \frac{dx}{dx} = \frac{d \sin y}{dx} = \frac{d \sin y}{dy}\frac{dy}{dx} = \cos y \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$$

Evaluate the indefinite integral

$$\int \frac{dx}{\sqrt{9-4x^2}}$$

$$= \frac{1}{3}\int \frac{dx}{\sqrt{1-\left(\frac{2}{3}x\right)^2}}$$

Now $\frac{d}{dx}\left(\sin^{-1}\left(\frac{2}{3}x\right)\right) = \frac{1}{\sqrt{1-\left(\frac{2}{3}x\right)^2}} \cdot \frac{2}{3}$

from 1st part above + ch. Rule.

$$\Rightarrow \frac{d}{dx}\left(\frac{1}{2}\sin^{-1}\left(\frac{2}{3}x\right)\right) = \frac{1}{3}\cdot\frac{1}{\sqrt{1-\left(\frac{2}{3}x\right)^2}}$$

$$\int = \boxed{\frac{1}{2}\sin^{-1}\left(\frac{2}{3}x\right) + C}$$

Q7]... **[14 points]** Evaluate the following integrals.

$$\int \frac{e^x\,dx}{1+e^{2x}}$$

Let $u = e^x$ $\quad e^{2x} = u^2$ $\quad du = e^x dx$

$$\int = \int \frac{du}{1+u^2} = \tan^{-1}(u) + c$$

$$\int = \boxed{\tan^{-1}(e^x) + C}$$

$$\int_e^{e^2} \frac{\ln(x)}{x}\,dx$$

let $u = \ln(x)$ $\quad du = \frac{1}{x}dx$

$x = e \Rightarrow u = \ln(e) = 1$

$x = e^2 \Rightarrow u = \ln(e^2) = 2$

$$\int = \int_1^2 u.du = \left.\frac{u^2}{2}\right]_1^2$$

$$= \frac{4}{2} - \frac{1}{2} = \boxed{\frac{3}{2}}$$