2) \[ \text{Cylindrical shell circumference} = 2\pi x \]

\[ V = \int_0^\pi 2\pi x \sin(x^2) \, dx \]

Let \( u = x^2 \) then \( du = 2x \, dx \)

For \( x = \pi \), \( u = \pi^2 \)

For \( x = 0 \), \( u = 0 \)

\[ \therefore \quad V = \int_0^\pi 2\pi \sin u \, du = \pi \int_0^\pi \sin u \, du = \pi [\sin u]_0^\pi = \pi [\sin \pi - \sin 0] = \pi [0 - 0] = 0 \]

For slicing, we need to find the local maximum point of the given function. When we find the local maximum we separate the curve \( \sin(x^2) = y \) into two pieces as \( x = f_1(y) \) and \( x = f_2(y) \) which are on the right hand side and left hand side on the graph respectively.

Hence \( V = \int_0^b [f_1(y)]^2 - [f_2(y)]^2 \, dy \). Using shells is definitely preferable to slicing.
16) \[ V = \int_{-2}^{1} 2\pi (-x) \cdot x^2 \, dx = 2\pi \left[ \frac{-1}{4} x^4 \right]_{-2}^{1} = \frac{13\pi}{2}. \]

20) \[ V = \int_{0}^{1} 2\pi (y+1) \left( y^2 - y^3 \right) \, dy = 2\pi \int_{0}^{1} (y^2 + y^3 - y^3 - y^4) \, dy = \frac{29\pi}{30} \]

38) \[ V = \int_{1}^{2} 2\pi x (-x^2 + 3x - 2) \, dx = 2\pi \int_{1}^{2} (-x^3 + 3x^2 - 2x) \, dx = \frac{\pi}{2} \]

by using shells.

40) By washer method
\[ V = \int_{1}^{2} \pi \left[ (2-y)^2 - (2-(1-y^4))^2 \right] \, dy = 2\pi \int_{1}^{2} \left[ 4 - (1+y^4) \right] \, dy \text{ (by symmetry)} \]
\[ = \frac{224\pi}{45} \]
42) By shells method

\[ V = \int_0^2 2\pi y \left[ \sqrt{1-(y-1)^2} - (-\sqrt{1-(y-1)^2}) \right] dy \]

\[ = 2\pi \int_0^2 y \cdot 2 \sqrt{1-(y-1)^2} dy \]

Let \( u = y - 1 \) then \( du = dy \) hence

\[ = 4\pi \int_{u=0}^{u=1} u \sqrt{1-u^2} du + 4\pi \int_{u=-1}^{u=1} \frac{1}{2} \sqrt{1-u^2} du \]

\[ \int_{-1}^{1} u \sqrt{1-u^2} du = 0 \text{ since its integrand is an odd function} \]

\[ \int_{-1}^{1} \sqrt{1-u^2} du = \frac{\pi}{2} \text{ since it is the area of a semicircle or radius 1} \]

hence

\[ V = 4\pi \cdot 0 + 4\pi \cdot \frac{\pi}{2} = 2\pi^2 \]

44) \[ V = \int_{r-R}^{r+R} 2\pi x \cdot 2 \sqrt{r^2-(x-R)^2} dx \]

Let \( u = x - R \) then \( du = dx \) so

\[ V = \int_{-r}^{r} 4\pi (u+R) \sqrt{r^2-u^2} du = 4\pi R \int_{-r}^{r} \sqrt{r^2-u^2} du + 4\pi \int_{-r}^{r} u \sqrt{r^2-u^2} du \]

\[ \int_{-r}^{r} \sqrt{r^2-u^2} du = \frac{1}{2} \pi r^2 \text{ since it is the area of a semicircle with radius } r \]

\[ \int_{-r}^{r} u \sqrt{r^2-u^2} du = 0 \text{ since the integrand is an odd function} \]
So
\[ V = 4\pi R \left( \frac{1}{2} \pi r^2 \right) + 4\pi R \cdot 0 = 2\pi R \cdot r^2 \]

By symmetry
\[
\text{inner radius} \quad V = 2 \int_0^R 2\pi rh \, dx = 2 \int_0^R 2\pi x \sqrt{R^2 - x^2} \, dx
\]
\[
= 4\pi \int_0^R \sqrt{R^2 - x^2} \, dx
\]
\[
= 4\pi \int_0^R \sqrt{R^2 - x^2} \, dx
\]

Let \( u = R^2 - x^2 \) then \( du = -2x \, dx \). \( \frac{dx}{-2} = \frac{du}{u} \)
\[
= 4\pi \int_{x=0}^{x=R} \frac{u^{1/2}}{2} \, du = -2\pi \left[ \frac{2}{3} u^{3/2} \right]_{x=0}^{x=R}
\]
\[
= -2\pi \left[ \frac{2}{3} (R^2 - x^2)^{3/2} \right]_{x=0}^{x=R}
\]
\[
= \frac{4}{3} \pi (R^2 - r^2)^{3/2}
\]

By Pythagorean theorem,
\[ R^2 = \left( \frac{1}{2} h \right)^2 + r^2 \]
so the volume of the napkin ring is \( \frac{4}{3} \pi \left( \frac{1}{2} h \right)^3 = \frac{1}{6} \pi h^3 \)
which is independent of both \( R \) and \( r \), so the amount of wood in a ring of height \( h \) is the same regardless of the size of the sphere used.
\[ V_{\text{napkin ring}} = V_{\text{sphere}} - V_{\text{cylinder}} - 2V_{\text{cap}} \]

\[ = \frac{4}{3} \pi R^3 - \pi R^2 h - 2 \pi h \left( R^2 - \frac{1}{2} R h \right)^2 \]

where height of cap is \( R - \frac{1}{2} h \).