Q1]. . . [13 points] Use logarithmic differentiation to compute the derivative $f'(x)$ of the following function

$$f(x) \;=\; x^{\sqrt{x}}$$

Take logs of the equation $y = x^{\sqrt{x}}$ to get $\ln y = \ln(x^{\sqrt{x}}) = \sqrt{x}\ln(x)$. Differentiate this expression with respect to x to get

$$\frac{1}{y}y' \;=\; \frac{1}{2}x^{-1/2}\ln(x) + \sqrt{x}\frac{1}{x} \;=\; \frac{\ln(x)+2}{2\sqrt{x}}\,.$$

Thus,

$$y' \;=\; x^{\sqrt{x}}\frac{(\ln(x)+2)}{2\sqrt{x}}\,.$$

Compute the second derivative $g''(x)$ of the following function

$$g(x) \;=\; \int_1^x \ln(t+1)\,dt$$

By the Fundamental Theorem of Calculus we have

$$g'(x) \;=\; \ln(x+1)$$

and differentiating this again gives

$$g''(x) \;=\; \frac{1}{x+1}\,.$$

Q2]. . . [13 points] Write down the "cylindrical shell method" formula for the volume obtained by rotating the region R about the y-axis. Here R is the region below the graph $y = f(x)$ between $x = a$ and $x = b$.

$$V \;=\; 2\pi\int_a^b x f(x)\,dx$$

Compute the volume obtained by rotating the region R about the y-axis. R is the region bounded by the graph $y = 1/(x\sqrt{1-x^2})$, the x-axis, and the lines $x = 1/2$ and $x = \sqrt{3}/2$.

$$V \;=\; 2\pi\int_{1/2}^{\sqrt{3}/2}\frac{x}{x\sqrt{1-x^2}}\,dx \;=\; 2\pi\int_{1/2}^{\sqrt{3}/2}\frac{dx}{\sqrt{1-x^2}} \;=\; 2\pi\sin^{-1}(x)|_{1/2}^{\sqrt{3}/2}$$

Since $\sin^{-1}(\sqrt{3}/2) = \pi/3$ and $\sin^{-1}(1/2) = \pi/6$ we get $V = 2\pi(\pi/6) = 3\pi^2$.

Q3]... [12 points] Evaluate the following indefinite integrals.

$$\int \frac{x\,dx}{1+x^2}$$

Let $u = 1 + x^2$, so that $du = 2xdx$. The integral becomes (by the method of substitution)

$$\frac{1}{2}\int \frac{du}{u} \;=\; \frac{1}{2}\ln|u| + C \;=\; \frac{1}{2}\ln(1+x^2) + C\,.$$

$$\int \frac{x\,dx}{1+x^4}$$

Let $u = x^2$ so that $du = 2xdx$ and the integral becomes (by substitution)

$$\frac{1}{2}\int \frac{du}{1+u^2} \;=\; \frac{1}{2}\tan^{-1}(u) + C \;=\; \frac{1}{2}\tan^{-1}(x^2) + C\,.$$

Q4]... [12 points] A bucket is raised vertically from the ground at a speed of 2 m/min using a rope of negligible weight (ignore the weight of the rope in your calculations). The bucket weighs 1 kg, and is filled with 15 kg of water at the start. As it is being pulled up, the bucket leaks water at a constant rate of 1 kg per minute. Write down an integral for the **work** done in raising the leaking bucket to a height of 10 m. Show how you arrived at your answer. You do not need to evaluate the integral.

Let t denote time (in minutes). Total time is $10/2 = 5$ minutes. At time t the mass of the bucket is $(1 + 15) - 1t$ kg, or simply $(16 - t)$ kg.

Divide the time interval into n equal width subintervals. During the time interval from t_{i-1} to t_i we can take t_i^* as the approximate time. The mass of the bucket is $16 - t_i^*$ kg. Gravity exerts a **force** of $(16 - t_i^*)g$ Newtons on the bucket of water during this time. Here g is the acceleration due to gravity on the earth's surface. This mass is moved a distance of $2\Delta t$ m during this interval.

Thus the approximate work done during this interval is simply the product $(16 - t_i^*)g(2\Delta t)$. Adding all these contributions to work done gives a sum

$$\sum_{i=1}^{n}(16 - t_i^*)g(2\Delta t)\,.$$

Taking the limit as $n \to \infty$ gives an exact expression for the work done; namely

$$W \;=\; \int_0^5 (16-t)g2\,dt \;=\; 2g\int_0^5 (16-t)\,dt\,.$$

Note, if you worked with displacement x instead of time t you would get the integral

$$W \;=\; g\int_0^{10} (16 - x/2)\,dx\,.$$

If you had to evaluate these integrals, you would use a numerical value for g, the acceleration due to gravity. This is usually 9.8 m/s^2. However, since our problem is stated in minutes instead of seconds, you would convert 9.8 m/s^2 to $9.8(60)^2$ m/min^2, and use $g = 9.8(60)^2$. You might use a different number if this bucket of water were being raised on the surface of a different planet.