Math 2443–002 Div, Grad and Curl Spring 2000

Nabla: The vector differential operator ∇ is pronounced *nabla*, and is defined to be

$$\nabla \;=\; \frac{\partial}{\partial x}\,\hat{\mathbf{i}} + \frac{\partial}{\partial y}\,\hat{\mathbf{j}} + \frac{\partial}{\partial z}\,\hat{\mathbf{k}}$$

Grad: ∇ can act on a function (scalar field) f to give a vector field called the *gradient of* f.

$$\nabla f \;=\; \frac{\partial f}{\partial x}\,\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\,\hat{\mathbf{j}} + \frac{\partial f}{\partial z}\,\hat{\mathbf{k}} \;=\; \langle\, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\,\rangle$$

Recall that ∇f is always perpendicular to the level surfaces of f, points in the direction of maximum rate of increase of f, and that $|\nabla f|$ is this maximum rate of increase.

Div: The operator ∇ can act on a vector field $\mathbf{F}$ in the same way as a dot product. This gives us a scalar field (or function) called the *divergence of* $\mathbf{F}$ which is defined by

$$div\mathbf{F} \;=\; \nabla \cdot \mathbf{F} \;=\; \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

where $\mathbf{F} = \langle P, Q, R\rangle$.

Curl: The operator ∇ can act on a vector field $\mathbf{F}$ in the same was as a cross product. This gives another vector field called the *curl of* $\mathbf{F}$ which is defined by

$$curl\mathbf{F} \;=\; \nabla \times \mathbf{F} \;=\; \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \;=\; \langle\, R_y - Q_z,\, P_z - R_x,\, Q_x - P_y\,\rangle$$

Laplace operator: The Laplace operator acts on a scalar field or function and is defined to be *div grad*

$$\nabla^2 f \;=\; \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

Two interpretations of Green's Theorem: The two equations below are versions of Green's theorem. They help us gain an intuitive understanding of *div* and *curl*.

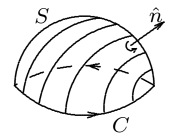
$$\int_C (\mathbf{F}\cdot\hat{\mathbf{T}})\,ds \;=\; \int_C \mathbf{F}\cdot d\mathbf{r} \;=\; \int\!\!\int_D curl(\mathbf{F})\cdot\hat{\mathbf{k}}\,dA \qquad \text{Tangential components,}$$

and

$$\int_C (\mathbf{F}\cdot\hat{\mathbf{n}})\,ds \;=\; \int\!\!\int_D div(\mathbf{F})\,dA \qquad \text{Normal components.}$$

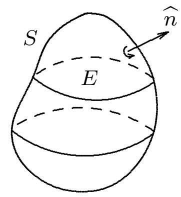
Stokes Theorem: Let S be an oriented, piecewise-smooth surface bounded by a positively oriented, simple, closed, piecewise-smooth curve C. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region of $\mathbf{R}^3$ which contains S. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S curl(\mathbf{F}) \cdot d\mathbf{s}$$



Divergence Theorem: Let E be a simple solid region in $\mathbf{R}^3$ whose boundary surface S has a positive (outward) orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region of $\mathbf{R}^3$ which contains E. Then

$$\iint_S \mathbf{F} \cdot d\mathbf{s} = \iiint_E div(\mathbf{F})\, dv$$



Physical Interpretations: Let $\mathbf{F}$ be the velocity vector field of a fluid flow. Let $P_0 = (x_0, y_0, z_0)$ be a point in the fluid flow, and $\mathbf{n}$ be a unit vector based at P_0. Then

$$curl(\mathbf{F})(P_0) \cdot \mathbf{n}$$

measures the rotating effect of the fluid about the axis $\mathbf{n}$. This rotating effect is greatest about an axis which points in the direction of $curl(\mathbf{F})(P_0)$.

Also $div(\mathbf{F})(P_0)$ measures the net rate of outward flux per unit volume at P_0.