Q1]... State the second derivative test for functions of two variables.

**Ans:** Let \((a, b)\) satisfy \(f_x(a, b) = 0\) and \(f_y(a, b) = 0\). Define

\[
D(x, y) = (f_{xx})(f_{yy}) - (f_{xy})^2
\]

- If \(D(a, b) > 0\) and \(f_{xx}(a, b) > 0\), then \((a, b)\) is a local minimum point.
- If \(D(a, b) > 0\) and \(f_{xx}(a, b) < 0\), then \((a, b)\) is a local maximum point.
- If \(D(a, b) < 0\) and \(f_{xx}(a, b) > 0\), then \((a, b)\) is neither a local max nor a local min point [Saddle].
- (If \(D(a, b) = 0\), the test is inconclusive.)

Q2]... Find and test the critical points of the function

\[
f(x, y) = xye^{-(x^2+y^2)/2}.
\]

**Soln:** We compute the first derivatives using the product and chain rules.

\[
f_x = (1 - x^2)ye^{-(x^2+y^2)/2} \quad f_y = (1 - y^2)xe^{-(x^2+y^2)/2}.
\]

Since \(e\) to any power is always positive, we see that \(f_x = 0 = f_y\) if and only if \((1 - x^2)y = 0 = (1 - y^2)x\), and that these equations are true if and only if \((x, y)\) is one of the following five points: \((0,0), (1,1), (1,-1), (-1,1), (-1,-1)\).

Now, the second derivatives work out to be

\[
f_{xx} = -2xye^{-(x^2+y^2)/2} - xy(1 - x^2)e^{-(x^2+y^2)/2},
\]

and

\[
f_{yy} = -2xye^{-(x^2+y^2)/2} - xy(1 - y^2)e^{-(x^2+y^2)/2},
\]

and

\[
f_{xy} = (1 - x^2)(1 - y^2)e^{-(x^2+y^2)/2}.
\]

Thus, we can evaluate the second derivatives at the critical points to get.

- \(D(0,0) = 0^2 - 1^2 = -1 < 0\) implies a saddle point at \((0,0)\).
- \(D(-1,1) = D(1,-1) = (2e^{-1})^2 - 0^2 > 0\), and \(f_{xx}(-1,1) = f_{xx}(1,-1) = 2e^{-1} > 0\) implies a local minimum at \((-1,1)\) and at \((1,-1)\).
- \(D(-1,-1) = D(1,1) = (-2e^{-1})^2 - 0^2 > 0\), and \(f_{xx}(-1,-1) = f_{xx}(1,1) = -2e^{-1} < 0\) implies a local maximum at \((-1,-1)\) and at \((1,1)\).