MATH 2924

EXAM 3 SOLUTIONS

Many people missed the fact that there is a (2n-1) term in the numerator which cancels with the (2n-1) term in the denominator.

#1 $$\sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots\cdots(2n-1)}{5^n\cdot n!}$$

$$a_n = \frac{1\cdot 3\cdot 5\cdot\cdots\cdots(2n-1)}{5^n\cdot n!}, \quad a_{n+1} = \frac{1\cdot 3\cdot 5\cdots\cdots(2n-1)(2n+1)}{5^{n+1}\cdot(n+1)!}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{1\cdot 3\cdot 5\cdots(2n-1)(2n+1)}{5^{n+1}\cdot(n+1)!}\cdot\frac{5^n\cdot n!}{1\cdot 3\cdot 5\cdots(2n-1)}\right|$$

$$= \lim_{n\to\infty}\left|\frac{2n+1}{5(n+1)}\right|$$

$$= 2/5 < 1$$

$\Rightarrow$ the series converges by Ratio test $\square$

#2 $$\sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{7n+1}}$$

Consider $\displaystyle\sum_{n=0}^{\infty}\left|\frac{(-1)^n}{\sqrt{7n+1}}\right| = \sum_{n=0}^{\infty}\frac{1}{\sqrt{7n+1}} = \sum_{n=0}^{\infty} a_n$ (say)

let $b_n = \dfrac{1}{\sqrt{n}}$

we know by p-series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ diverges

$$\lim_{n\to\infty}\left|\frac{a_n}{b_n}\right| = \lim_{n\to\infty}\left|\frac{\sqrt{n}}{\sqrt{7n+1}}\right| = \frac{1}{\sqrt{7}} > 0$$

$\Rightarrow$ By $\sum a_n$ Limit comparision test

$\Rightarrow$ $\sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{7n+1}}$ is "NOT" Absolutely convergent

Lets check for convergence.

$a_n = \frac{1}{\sqrt{7n+1}}$, $7(n+1)+1 = 7n+8 > 7n+1$

$$\Rightarrow \frac{1}{7n+1} > \frac{1}{7n+8}$$

$$\Rightarrow \frac{1}{\sqrt{7n+1}} > \frac{1}{\sqrt{7n+8}}$$

$$\Rightarrow a_n > a_{n+1}$$

also $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{\sqrt{7n+1}} = 0$

$\Rightarrow$ By Alternating series test

$\sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{7n+1}}$ is convergent

$\Rightarrow$ the series is Conditionally convergent

$\boxtimes$

#3 $\sum_{n=1}^{\infty} \frac{(5x-4)^n}{n^3}$

$a_n = \frac{(5x-4)^n}{n^3}$, $a_{n+1} = \frac{(5x-4)^{n+1}}{(n+1)^3}$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(5x-4)^{n+1}}{(n+1)^3} \cdot \frac{n^3}{(5x-4)^n}\right|$

$= |5x-4| \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^3$

$= |5x-4|$

want $|5x-4| = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$

for convergence

$|5x-4| < 1 \Rightarrow |x - 4/5| < 1/5$

$\Rightarrow$ Radius of conv. $= 1/5$

$-1/5 < x - 4/5 < 1$

$\Rightarrow 3/5 < x < 1$

Check at end points: $x = 1$, $\sum_{n=1}^{\infty} \frac{(5-4)^n}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^3}$ } converges By p series

$x = 3/5$, $\sum_{n=1}^{\infty} \frac{(5(3/5)-1)^n}{n^3} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$ } converges By Alternating series test

$\Rightarrow$ Interval of convergence $= [3/5, 1]$

$\boxed{\#4}$ Taylor series for $f(x) = \cos x$ about $x = \pi/2$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n, \quad a = \pi/2$$

$$f(\pi/2) = 0$$

$$f'(x) = \frac{d}{dx}\cos x = -\sin x \Rightarrow f'(\pi/2) = -1$$

$$f''(x) = -\cos x \Rightarrow f^{(2)}(\pi/2) = 0$$

$$f^{(3)}(x) = \sin x \Rightarrow f^{(3)}(\pi/2) = 1$$

$$\Rightarrow f^{(n)}(\pi/2) = \begin{cases} 0 & \text{if } n \text{ is even} \\ (-1) & \text{if } n \text{ is } 1, 5, 9, \ldots \\ 1 & \text{if } n \text{ is } 3, 7, 11, \ldots \end{cases}$$

$$\Rightarrow f(x) = \frac{-1}{1!}(x-\pi/2) + \frac{1}{3!}(x-\pi/2)^3 + \frac{(-1)}{5!}(x-\pi/2)^5 + \frac{1}{7!}(x-\tfrac{\pi}{2})^7 + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{(x-\pi/2)^{2n+1}}{(2n+1)!}$$

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$\boxed{\#5}$ Estimate $\int_0^1 \cos x^2\, dx$ with in 0.01

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \Rightarrow \cos(x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n}}{(2n)!}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{(2n)!}$$

$$\int_0^1 \cos x^2\, dx = \int_0^1 \left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{(2n)!} \right) dx$$

$$= \sum_{n=\infty}^{\infty} \left[\int_0^1 (-1)^n \frac{x^{4n}}{(2n)!} dx \right]$$

$$= \sum_{n=0}^{\infty} \left[\frac{(-1)^n x^{(4n+1)}}{(2n)!(4n+1)} \right]_0^1$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+1)(2n)!}$$

You can write series out term by term here too. You do not have to write out the general term for full points.

$$S_0 = \frac{(-1)^0}{0! \cdot 1} = 1$$

difference $= \frac{1}{10} = 0.1$

$$S_1 = S_0 + \frac{(-1)}{5 \cdot 2!} = 1 - \frac{1}{10}$$

diff. $= \frac{1}{9 \cdot 24} < \frac{1}{100} = 0.01$

$$S_2 = S_1 + \frac{1}{((4 \cdot 2)+1)(4!)} = 1 - \frac{1}{10} + \frac{1}{9 \cdot 24}$$

$$\Rightarrow \int_0^1 \cos x^2 dx \approx 1 - \frac{1}{10} + \frac{1}{216}$$

The answer 1-1/10 is also within 1/216 < 0.01 of the precise value.

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