MATH 2924 - 010
MIDTERM - I SOLUTIONS

#1 (i) $y = (\sin x)^x - x^{\sin x}$

$$\frac{dy}{dx} = \frac{d}{dx}(\sin x)^x - \frac{d}{dx} x^{\sin x}$$

Note: $\log(a-b) \neq \log(a) - \log(b)$

in particular $\log((\sin x)^x - x^{\sin x}) \neq x\log(\sin x) - \sin x \log x$.

$y_1 = (\sin x)^x \Rightarrow \log y_1 = x \log(\sin x)$

$$\frac{1}{y_1}\frac{dy_1}{dx} = \log(\sin x) + \frac{x}{\sin x} \cdot \cos x$$

$$\therefore \frac{d(\sin x)^x}{dx} = (\sin x)^x \left[\log(\sin x) + \frac{x}{\sin x} \cdot \cos x\right]$$

$y_2 = x^{\sin x} \Rightarrow \log y_2 = \sin x \log x$

$$\frac{1}{y_2}\frac{dy_2}{dx} = \frac{\sin x}{x} + \cos x \log x$$

$$\therefore \frac{d}{dx}(x^{\sin x}) = x^{\sin x}\left(\frac{\sin x}{x} + \cos x \log x\right)$$

$$\Rightarrow \boxed{\frac{dy}{dx} = (\sin x)^x\left[\log(\sin x) + x\cot x\right] - x^{\sin x}\left(\frac{\sin x}{x} + \cos x \cdot \log x\right)}$$

(2) $y = \log_{\pi}(x) = \dfrac{\ln x}{\ln \pi}$

$\therefore \dfrac{dy}{dx} = \dfrac{1}{\ln \pi} \dfrac{d}{dx} \ln x$

$$\boxed{\frac{dy}{dx} = \frac{1}{x \ln \pi}}$$

#2 (1) to show: $\dfrac{d}{dx} \cosh x = \sinh x$

$\cosh x = \dfrac{e^{x} + e^{-x}}{2}$

$\dfrac{d}{dx} \cosh x = \dfrac{1}{2} \left[\dfrac{d}{dx} e^{x} + \dfrac{d}{dx} e^{-x} \right] = \dfrac{1}{2} (e^{x} - e^{-x})$

$= \sinh x$

(2) $y = \dfrac{T_0}{K} \cosh\left(\dfrac{K}{T_0} x\right)$

$\dfrac{dy}{dx} = \sinh\left(\dfrac{K}{T_0} x\right)$

$\dfrac{d^2y}{dx^2} = \dfrac{K}{T_0} \cosh\left(\dfrac{K}{T_0} x\right)$

$= \dfrac{K}{T_0} \sqrt{1 + \sinh^2\left(\dfrac{K}{T_0} x\right)} = \dfrac{K}{T_0} \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}$

[use: $\cosh^2 x - \sinh^2 x = 1$]

#3 (1) half life = 1200 years

let $N(0) = 100$

$\Rightarrow N(1200) = 50$

exponential decay formula: $N(t) = N(0)e^{kt}$

for $t = 1200$

$$50 = 100\, e^{k \cdot 1200}$$

$$0.5 = e^{k \cdot 1200}$$

$$\therefore k = \frac{\ln(0.5)}{1200}$$

to find $N(20)$

$$N(20) = N(0) \cdot e^{k \cdot 20} = 100 \cdot e^{\frac{\ln(0.5)}{1200} \cdot 20}$$

$$= 100 \cdot e^{\frac{\ln(0.5)}{60}}$$

$$= \boxed{100 \cdot (0.5)^{1/60}}$$

(2) let 20% of original sample decayed in t years

$N(t) = 80$

$$\therefore 80 = 100 \cdot e^{\frac{\ln(0.5)}{1200} \cdot t}$$

$$\therefore \boxed{t = 1200 \cdot \frac{\ln(0.8)}{\ln(0.5)} \text{ years}}$$

$\boxed{\#4}$ (1) $\displaystyle\int \frac{dx}{x\ln x}$ $\quad u = \ln x$

$\Rightarrow du = \frac{1}{x}dx$

$$= \int \frac{du}{u} = \ln|u| + C$$

$$= \ln|\ln x| + C$$

(2) $\displaystyle\int \frac{dx}{7+(x-1)^2} = \frac{1}{7}\int \frac{dx}{1+\left(\frac{x-1}{\sqrt{7}}\right)^2}$

let $u = \dfrac{x-1}{\sqrt{7}}$

$du = \dfrac{1}{\sqrt{7}}dx$

$$= \frac{1}{\sqrt{7}}\int \frac{du}{1+u^2}$$

$$= \frac{1}{\sqrt{7}}\tan^{-1}(u) + C = \boxed{\frac{1}{\sqrt{7}}\tan^{-1}\left(\frac{x-1}{\sqrt{7}}\right) + C}$$