$\sqrt[3]{2}$ is irrational.

Proof : We argue by contradiction. Suppose $\sqrt[3]{2}$ is rational. This means

$$\sqrt[3]{2} = p/q \text{ , for some } p, q \in \mathbb{N} \quad \text{—} (*)$$

By the Least Principle, there is an expression $(*)$ where q is the least possible denominator.

Now $\frac{p^3}{q^3} = (\sqrt[3]{2})^3 = 2$

& so $p^3 = 2q^3$. ——— $[\dagger]$

Both sides of Equation $[\dagger]$ are integers, and the RHS is clearly even.

$\Rightarrow p^3$ is even

$\Rightarrow p$ is even (proven in class)

$\Rightarrow p = 2k$ for some $k \in \mathbb{N}$.

$\Rightarrow [\dagger]$ becomes

$(2k)^3 = 2q^3$

$\Rightarrow 8k^3 = 2q^3$

$\Rightarrow 4k^3 = q^3$

$\rightarrow$ Now LHS of this is even.

$\Rightarrow q^3$ even

$\Rightarrow q$ even (proven in class)

$\Rightarrow q = 2l$ for some $l \in \mathbb{N}$.

Finally

$$\sqrt[3]{2} = p/q = \frac{(2k)}{(2\ell)} = \frac{k}{\ell}$$

gives another expression of the form (*) for $\sqrt[3]{2}$ with $\ell = q/2 < q$. This is a contradiction.

Therefore, $\sqrt[3]{2}$ is irrational

The sum of a rational and an irrational number is irrational.

<u>Proof</u> Let x be rational and y irrational; we want to prove that $x+y$ is irrational.

We argue by contradiction. Suppose $x+y$ is rational. This means

$$x+y = p/q \qquad \text{for some } p, q \in \mathbb{Z},\ q \neq 0.$$

But x is rational, and so $x = a/b$ for some $a, b \in \mathbb{Z}$, $b \neq 0$.

Substituting for x gives

$$\frac{a}{b} + y = p/q$$

Thus $$y = \frac{p}{q} - \frac{a}{b} = \frac{pb - aq}{qb}$$

this is a ratio of two integers, and denominator $\neq 0$.

$\Rightarrow$ y is rational, a contradiction.

Therefore, the assumption that $x+y$ is rational is FALSE.

$\Rightarrow$ $x+y$ is irrational $\boxtimes$

The product of an irrational number and a non-zero rational number is irrational.

Proof Let x be irrational and $y \neq 0$ be rational. We argue that xy is irrational by contradiction.

Assume xy is rational. Therefore, $xy = p/q$ for some $p, q \in \mathbb{Z}$, $q \neq 0$.

We know $0 \neq y$ is rational. Therefore $y = a/b$ for some $a, b \in \mathbb{Z}$ with $b \neq 0$ and $a \neq 0$.

Substituting for y gives

$$x \frac{a}{b} = \frac{p}{q}$$

$$\Rightarrow \quad x = \frac{bp}{aq}$$ ---is a ratio of integers, with denominator $\neq 0$ (since $a \neq 0$, $q \neq 0$).

$\Rightarrow$ x is rational, a contradiction.

Therefore, the assumption that xy is rational is FALSE.

$\Rightarrow$ xy is irrational. ∎