Prop.

$$f : \left(-\pi/2, \pi/2\right) \longrightarrow \mathbb{R}$$
$$: x \longmapsto \tan(x)$$

is <u>injective</u>.

Proof We know from Calc I that

$f'(x) = \sec^2(x)$ is well-defined (no $\div$ by 0) and is positive on $(-\pi/2, \pi/2)$

Calc I $\Rightarrow$ $f(x)$ is increasing on the interval $(-\pi/2, \pi/2)$.
$\uparrow$
(M.V.T.)

Now suppose $x_1, x_2 \in (-\pi/2, \pi/2)$ are not equal

$$x_1 \neq x_2 \Rightarrow \left.\begin{cases} x_1 < x_2 \Longrightarrow f(x_1) < f(x_2) \\ \underline{\text{OR}} \qquad \updownarrow f \text{ increasing} \\ x_2 < x_1 \Longrightarrow f(x_2) < f(x_1) \end{cases}\right\} \Rightarrow f(x_1) \neq f(x_2).$$

We have seen (in both cases) that $(x_1 \neq x_2) \rightarrow (f(x_1) \neq f(x_2))$

Thus f is injective. $\blacksquare$

Prop:

$$f : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$$
$$: (m, n) \longmapsto 5^m 7^n$$

is injective.

Proof We will show for any (m_1, n_1) and (m_2, n_2) in $\mathbb{N} \times \mathbb{N}$;

$$\left(f(m_1, n_1) = f(m_2, n_2)\right) \longrightarrow \left((m_1, n_1) = (m_2, n_2)\right).$$

$$f(m_1, n_1) = f(m_2, n_2) \Rightarrow 5^{m_1} 7^{n_1} = 5^{m_2} 7^{n_2}$$

But we have an integer which has a prime decomp. of the form $5^{m_1} 7^{n_1}$ and a <u>second</u> prime decomp. of the form $5^{m_2} 7^{n_2}$. By uniqueness of prime decompositions (F.T.A.) we conclude that

$$m_1 = m_2 \quad \text{and} \quad n_1 = n_2$$

Thus $(m_1, n_1) = (m_2, n_2)$

& so f is injective.

$\boxed{/\!/}$