Hwk 15

Q1 Subgroups of Perm({1,2,3}) $\qquad |\text{Perm}(\{1,2,3\})| = 6$

So Lagrange's theorem says subgroups must have size 1, 2, 3, 6 (factors of 6).

Also every subgroup must contain the identity 1, must be closed under products + inverses.

$\boxed{|H|=1}$ only one, namely $H_1 = \{1\}$

$\boxed{|H|=2}$ Only 3, namely
$$H_2 = \{1, (12)\}$$
$$H_3 = \{1, (13)\}$$
$$H_4 = \{1, (23)\}$$

Note $\{1, (123)\}$ ← must also contain $(123)^2 = (123)^{-1} = (132)$ if it is to be a subgroup!

$\boxed{|H|=3}$ $\{1, (123), (132)\} = H_5$ only one!

eg $\{1, (12), (13)\}$ is not a subgroup because it would have to contain $(12)(13) = (132)$ and then also contain $(132)^{-1} = (123)$

$\boxed{|H|=6}$ $H_6 = \{1, (12), (13), (23), (123), (132)\} = \text{Perm}(\{1,2,3\})$

Q2

$$\left.\begin{aligned}(1234)^2 &= (13)(24) \neq 1\\ (1234)^3 &= (1432) \neq 1\\ (1234)^4 &= 1\end{aligned}\right\} \text{ order } (1234) = 4.$$

$$\left.\begin{aligned}(12345)^2 &= (13524) \neq 1 \qquad & (12345)^4 &= (15432) \neq 1\\ (12345)^3 &= (14253) \neq 1 & (12345)^5 &= 1\end{aligned}\right\} \Rightarrow \text{order}(12345) = 5$$

$$\left.\begin{aligned}((123)(45))^2 &= (123)^2(45)^2 = (132) \neq 1\\ ((123)(45))^3 &= (123)^3(45)^3 = (45) \neq 1\\ ((123)(45))^4 &= (123)^4(45)^4 = (123) \neq 1\\ ((123)(45))^5 &= (123)^5(45)^5 = (132)(45) \neq 1\\ ((123)(45))^6 &= (123)^6(45)^6 = 1\end{aligned}\right\} \Rightarrow \text{order}((123)(45)) = 6$$

$$((12)(34))^2 = 1 \qquad \Rightarrow \quad \text{order}((12)(34)) = 2$$

$$(12)^2 = 1 \qquad \Rightarrow \quad \text{order}((12)) = 2$$

$$\left.\begin{aligned}(123)^2 &= (132) \neq 1\\ (123)^3 &= 1\end{aligned}\right\} \Rightarrow \quad \text{order}((123)) = 3$$

Note: 5, 6, 2, 3 all divide $|\text{Perm}(\{1,2,3,4,5\})| = 120$.

Q3

$\operatorname{ord}(0) = 1$ $\qquad 0 = 0$

$\operatorname{ord}(1) = 6$ $\qquad \underbrace{1+1+\cdots+1}_{6} = 6 \equiv 0 \pmod 6$

& $\not\equiv 0 \pmod 6$ for fewer than 6 terms.

$\operatorname{ord}(2) = 3$ $\qquad 2+2 = 4 \not\equiv 0 \pmod 6$

$2+2+2 = 6 \equiv 0 \pmod 6$

$\operatorname{ord}(3) = 2$ $\qquad 3+3 \equiv 0 \pmod 6$

$\operatorname{ord}(4) = 3$ $\qquad 4+4 = 8 \equiv 2 \not\equiv 0 \pmod 6$

$4+4+4 = 12 \equiv 0 \pmod 6$

$\operatorname{ord}(5) = 6$ $\qquad 5+5 = 4 \not\equiv 0 \pmod 6$

$5+5+5 \equiv 9 \equiv 3 \not\equiv 0 \pmod 6$

$5+5+5+5 \equiv 8 \equiv 2 \not\equiv 0 \pmod 6$

$5+5+5+5+5 \equiv 7 \equiv 1 \not\equiv 0 \pmod 6$

$5+5+5+5+5+5 \equiv 6 \equiv 0 \pmod 6$

Note: orders are 1, 2, 3, 6, all of which divide

$$6 = |\mathbb{Z}_6|.$$

Q4

$\text{ord}(1) = 1$ $\qquad 1 \equiv 1 \pmod 7$

$\text{ord}(2) = 3$ $\qquad 2^2 = 4 \not\equiv 1 \pmod 7$

$2^3 = 8 \equiv 1 \pmod 7$

$\text{ord}(3) = 6$ $\qquad 3^2 = 9 \equiv 2 \pmod 7$

$3^3 \equiv 6 \equiv -1 \pmod 7$

skip $3^4, 3^5$ since $4, 5 \nmid 6$

$3^6 \equiv (3^3)^2 \equiv (-1)^2 \equiv 1 \pmod 7$

$\text{ord}(4) = 3$ $\qquad 4^2 = 16 \equiv 2 \pmod 7$

$4^3 \equiv 8 \equiv 1 \pmod 7$

$\text{ord}(5) = 6$ $\qquad 5^2 = 25 \equiv 4 \pmod 7$

$5^3 \equiv 20 \equiv -1 \pmod 7$

skip 4, 5 since $4, 5 \nmid 6$

$5^6 \equiv (-1)^2 \equiv 1 \pmod 7$

$\text{ord}(6) = 2$ $\qquad 6 \equiv -1 \not\equiv 0 \pmod 7$

$6^2 \equiv (-1)^2 \equiv 1 \pmod 7$

Note: Orders are 1, 2, 3, 6, all of which divide into $|\mathbb{Z}_7 - \{0\}| = 6$.

← Q5 was not asked →