$P(x)$: $x^2 \geq 0$ $\qquad U = \mathbb{R}$

(1) $(\forall x \in \mathbb{R})(x^2 \geq 0)$ This is TRUE

why?

There are 3 cases. CASE (1): $x > 0$. Then $x^2 > 0$ ... (pos)(pos) = (pos)

CASE (2): $x < 0$. Then $x^2 > 0$ ... (neg)(neg) = (pos)

CASE (3): $x = 0$. Then $x^2 = 0$

In all 3 cases $x^2 \geq 0$. $\blacksquare$

(2) $(\exists x \in \mathbb{R})(x^2 \geq 0)$ This is True.

why?

It suffices to produce a single example. (We know any number x will work from the argument above in part (1).) Eg.

$x = 2$ $\qquad 2^2 = 4 \geq 0$.

(3) Negation of (1). $\neg(\forall x \in \mathbb{R})(x^2 \geq 0) \equiv (\exists x \in \mathbb{R})(x^2 < 0)$.

This is FALSE. why? because of the "3 cases" argument in (1) above. We know squares are never < 0.

(4) Negation of (2). $\neg(\exists x \in \mathbb{R})(x^2 \geq 0) \equiv (\forall x \in \mathbb{R})(x^2 < 0)$.

This is False. Producing one counterexample will show that this universally quantified claim is false.

The example from (2) $\quad 2^2 = 4 \not< 0$ works.

$P(x)$: $x^2 \geq 2$ $\qquad U = \mathbb{R}$

---

(1) $(\forall x \in \mathbb{R})(x^2 \geq 2)$ False

Counterexample . $x=1$ $\quad x^2 = 1 \not\geq 2$.

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(2) $(\exists x \in \mathbb{R})(x^2 \geq 2)$ TRUE

One example suffices.

$x = 7$, $\quad x^2 = 49 \geq 2$.

---

(3) Negation of (1). $\quad \neg(\forall x \in \mathbb{R})(x^2 \geq 2) \equiv (\exists x \in \mathbb{R})(x^2 < 2)$

This is TRUE.

one example (eg $x=1$, $x^2 = 1 < 2$) suffices.

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(4) Negation of (2). $\quad \neg(\exists x \in \mathbb{R})(x^2 \geq 2) \equiv (\forall x \in \mathbb{R})(x^2 < 2)$

This is FALSE. One counterexample suffices. eg $x = 7$ $\quad x^2 = 49 \not< 2$.

$P(x,y): \quad x+y=5$ $\qquad U = \mathbb{R}$

---

(1) $(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})(x+y=5)$

This is <u>FALSE</u>. One counterexample, $x=4, y=7$

so $x+y = 4+7 = 11 \neq 5$, suffices.

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(2) <u>Negation of (1)</u>

$\neg(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})(x+y=5) \equiv \underline{(\exists x \in \mathbb{R})(\exists y \in \mathbb{R})(x+y \neq 5)}$

This is TRUE. One example suffices.

eg $x=4, y=7$ $\qquad x+y = 4+7 = 11 \neq 5$.

---

(3) $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(x+y=5)$.

This is TRUE. You might start with examples, but you should spot a simple algebra manipulation that every particular example shares, & you can give a general argument.

Given <u>any</u> $x \in \mathbb{R}$

choose $y = 5 - x$

Then $x+y = \cancel{x} + (5 - \cancel{x}) = 5$. $\blacksquare$

Negation of (3).

$$\neg(\forall x\in\mathbb{R})(\exists y\in\mathbb{R})(x+y=5) \equiv (\exists x\in\mathbb{R})(\forall y\in\mathbb{R})(x+y\neq 5).$$

This is false. It is claiming that there exists a magic number x with the property that no matter what number y I add to it, my answer will be $\neq 5$. But if I add $y=5-x$ to x then $x+(5-x)=5$, so no such magic number x exists.

Remark: when would $(\exists x\in\mathbb{R})(\forall y\in\mathbb{R})(P(x,y))$ be useful?

eg $$(\exists x\in\mathbb{R})(\forall y\in\mathbb{R})(x+y=y) \quad \text{——} (**)$$

The magic number x in this case exists! It is called 0, and the statement $(**)$ is just the claim that $\mathbb{R}$ has an additive identity element, viz. $x=0$.

---

(5) $(\exists x \in \mathbb{R})(\forall y \in \mathbb{R})(x + y = 5)$ (5)

This is FALSE. There is NO "magic" number x with this property "If I add <u>any</u> number y to x my answer is always 5".

eg. choose $y = 10 - x$

Then $x + y = x + (10 - x) = 10 \neq 5$.

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(6) <u>Negation of (5)</u>

$$\neg (\exists x \in \mathbb{R})(\forall y \in \mathbb{R})(x + y = 5) \equiv \underline{(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(x + y \neq 5)}.$$

This is TRUE.

We have seen a strategy above. Given <u>any</u> $x \in \mathbb{R}$ we simply choose $y = 10 - x$.

Then $x + y = x + (10 - x) = 10 \neq 5$.

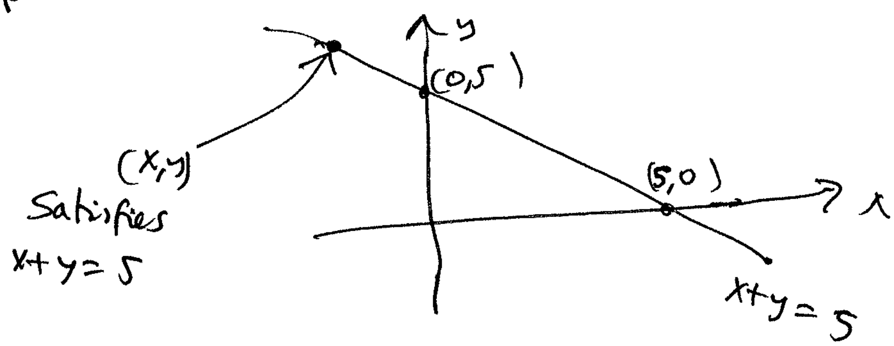
---

(7) $(\exists x \in \mathbb{R})(\exists y \in \mathbb{R})(x+y=5)$.

This is TRUE

eg $x=0$, $y=5$ $\qquad 0+5=5$ $\boxed{\checkmark}$

In fact there are infinitely many choices of pairs (x,y) --- If we plot these as points in the coordinate plane we would get a line



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(8) <u>Negation of (7)</u>

$$\neg(\exists x \in \mathbb{R})(\exists y \in \mathbb{R})(x+y=5) \equiv \underline{(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})(x+y \neq 5)}$$

This statement is FALSE. If I pick any point on the line in (7) above then its coordinates (x,y) satisfy $x+y=5$.

eg $x=5$, $y=0$ $\qquad 5+0=5$.

provides a <u>counterexample</u> to the claim.