MiD II (Solutions)

Q1]... **[10 points]** Compute the derivatives $\frac{dy}{dx}$ of the following functions y. Please show the steps of your work clearly.

- $y = (x^3 - 4x + 17)(x^2 + 2x^{\frac{3}{2}} - \sqrt[3]{x} + 9)$

$$y' = (3x^2 - 4x)(x^2 + 2x^{3/2} - x^{1/3} + 9)$$
$$+ (x^3 - 4x + 17)(2x + 3x^{1/2} - \tfrac{1}{3}x^{-2/3})$$

Used: Product + power rules

- $y = \frac{\sin(x)}{(x^2+5x+8)}$

$$y' = \frac{\cos(x)(x^2 + 5x + 8) - \sin(x)(2x+5)}{(x^2+5x+8)^2}$$

Used: quotient + power rules & $\frac{d}{dx} \sin x = \cos x$.

Q2]... [15 points] Compute the derivatives $\frac{dy}{dx}$ of the following functions y. Please show the steps of your work clearly.

- $y = \tan(\sqrt{3x^2+4})$

$$y' = \sec^2\left(\sqrt{3x^2+4}\right) \cdot \left(\frac{1}{2\sqrt{3x^2+4}}\right) \cdot (6x)$$

$u = \sqrt{3x^2+4}$

$v = 3x^2+4$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$$

Used: chain Rule (twice), power rules, + $\frac{d}{d\theta}\tan\theta = \sec^2\theta$.

- y is given implicitly by the equation

$$y\sin(x^2) = x\sin(y^2)$$

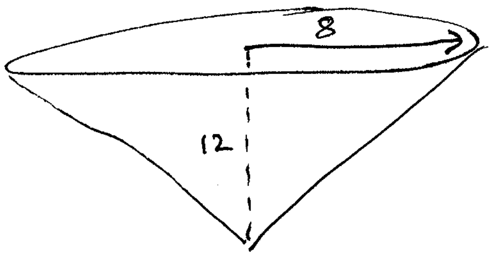
$$y'\sin(x^2) + y\cos(x^2)\cdot 2x = 1\cdot\sin(y^2) + x\cdot\cos(y^2)\cdot 2y\frac{dy}{dx}$$

$$y'\left(\sin(x^2) - 2xy\cos(y^2)\right) = \sin(y^2) - 2xy\cos(x^2)$$

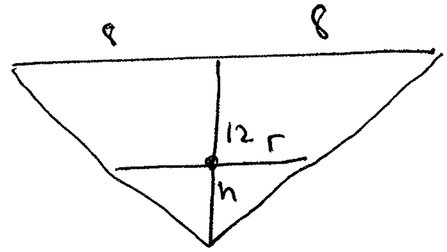
$$y' = \frac{\sin(y^2) - 2xy\cos(x^2)}{\sin(x^2) - 2xy\cos(y^2)}$$

Used: chain Rule, implicit diff., power rules, product rule and $\frac{d}{d\theta}\sin\theta = \cos\theta$.

Q3]...[15 points] A tank in the shape of an inverted cone is 16 ft across at the top and is 12 ft deep. Water is pouring into the tank at a rate 20 ft^3/min. Find how fast the height of the water is rising when the water is 6 ft deep. Show all the steps of your work.



CROSS $\longrightarrow$ SECTION



Similar Δ's $\quad \frac{r}{8} = \frac{h}{12}$

$$r = \frac{8}{12}h = \frac{2}{3}h$$

$$V = \frac{1}{3}(\pi r^2)h = \frac{1}{3}\pi\left(\frac{2}{3}h\right)^2 h = \frac{4\pi}{27}h^3$$

$$\Rightarrow \frac{dV}{dt} = \frac{d}{dt}\left(\frac{4\pi}{27}h^3\right) \underset{\substack{\uparrow \\ \text{Ch. Rule}}}{=} \frac{d}{dh}\left(\frac{4\pi}{27}h^3\right)\frac{dh}{dt}$$

$$\Rightarrow \frac{dV}{dt} = \frac{4\pi}{9}h^2\frac{dh}{dt}$$

At $h = 6 = (2)(3)$ we get ....

$$20 = \frac{4\pi}{\cancel{9}}\left((2)\cancel{(3)}\right)^{\cancel{2}}\frac{dh}{dt}$$

$$5\ \cancel{20} = \cancel{4}\pi(4)\frac{dh}{dt}$$

$$\boxed{\frac{dh}{dt} = \frac{5}{4\pi}\ \text{ft/min}}$$

We are told $\frac{dV}{dt} = 20$

We are asked for $\frac{dh}{dt}$ when $h = 6$.

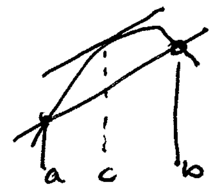
Q4]. . . [15 points]

- State the Mean Value Theorem.

Suppose f is differentiable on (a,b) & continuous on $[a,b]$.

There exists a number c in (a,b) so that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$



- Suppose that $f'(x) > 0$ for all x in an interval I. Show that the function $f(x)$ is increasing on I.

Given $x_1 < x_2$ in I

we know (MVT) that $\dfrac{f(x_2)-f(x_1)}{x_2-x_1} = f'(c)$ for some c between x_1 & x_2

But we are told $f'(c) > 0$.

$\Rightarrow$ fraction > 0 & Denominator $(x_2-x_1) > 0$

$\Rightarrow$ Numerator $f(x_2)-f(x_1) > 0$

$\Rightarrow$ $f(x_1) < f(x_2)$ $\Rightarrow$ f increasing.

- Suppose that f is differentiable everywhere and that $f' > 3$. If $f(-2) = 1$ find a lower bound for $f(3)$? Please show how you arrived at your answer.

$$\frac{f(3)-f(-2)}{3-(-2)} = f'(c) \quad \text{for some } c \text{ in } (-2,3)$$

MVT

told $f' > 3$

> 3

$\Rightarrow \dfrac{f(3)-1}{5} > 3$ $\Rightarrow f(3)-1 > 15$

$\Rightarrow \boxed{f(3) > 16}$

Q5]... **[15 points]** Determine the absolute maximum and the absolute minimum values of the function $f(x) = x\sqrt{4-x^2}$ on the interval $[-1, 2]$. Show all the steps of your work.

$$f'(x) = 1.\sqrt{4-x^2} + x.\frac{1\cdot(-\cancel{2}x)}{\cancel{2}\sqrt{4-x^2}}$$

$$= \frac{4-x^2 - x^2}{\sqrt{4-x^2}}$$

C.P.'s : $f'(x) = 0$ when $4-2x^2 = 0$ $x^2 = 2$ $x = \pm\sqrt{2}$

Now $-\sqrt{2}$ is outside of our interval, so we only need to consider the critical point $\sqrt{2}$.

Test points End points : $-1, 2$

Critical points : $\sqrt{2}$

$$f(-1) = (-1)\sqrt{4-1} = -\sqrt{3}$$

$$f(\sqrt{2}) = \sqrt{2}\sqrt{4-2} = (\sqrt{2})(\sqrt{2}) = 2$$

$$f(2) = 2\sqrt{4-4} = 0$$

So Absolute max is 2, & it occurs at $x = \sqrt{2}$

and

Absolute min is $-\sqrt{3}$, and it occurs at $x = -1$.

Q6]. . . [10 points] Write down the linear approximation to the function $f(x) = x^9$ at the point 1. Show your work.

$$L(x) = f(a) + f'(a)(x-a)$$

In our case $f(x) = x^9$ & $a = 1$.

$f'(x) = 9x^8$ $f(1) = 1$ $f'(1) = 9(1)^8 = 9$

$$\boxed{L(x) = 1 + 9(x-1)}$$

Use the linear approximation you obtained above to estimate $(1.02)^9$.

$$\begin{aligned}(1.02)^9 &= f(1.02)\\ &\approx L(1.02)\\ &= 1 + 9(1.02-1)\\ &= 1 + 9(0.02)\\ &= 1 + 0.18\\ &= 1.18\end{aligned}$$

Q7]... **[20 points]** Find the following information for the given function, using its derivatives which are given as well.

$f(x) = \dfrac{x^2-4}{x^2-16}$	$f'(x) = \dfrac{-24x}{(x^2-16)^2}$	$f''(x) = \dfrac{24(3x^2+16)}{(x^2-16)^3}$

a) Find the domain of $f(x)$.

Domain of $f = (-\infty,-4) \cup (-4,4) \cup (4,\infty)$

(all real #'s except where denom $= 0$).

b) Find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. Does $f(x)$ have any horizontal asymptotes? Does it have any vertical asymptotes?

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left(\frac{x^2-4}{x^2-16}\right) = \lim_{x\to\infty}\left(\frac{1-\frac{4}{x^2}}{1-\frac{16}{x^2}}\right) = \frac{1-0}{1-0} = 1$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\left(\frac{x^2-4}{x^2-16}\right) = \lim_{x\to-\infty}\left(\frac{1-\frac{4}{x^2}}{1-\frac{16}{x^2}}\right) = \frac{1-0}{1-0} = 1$$

Horizontal Asymptote $\boxed{y=1}$

Vertical Asymptotes at $\boxed{x=4}$ and at $\boxed{x=-4}$

c) Determine the intervals where $f(x)$ is increasing and decreasing. Find any local maxima or minima.

$f'(x) = 0$ when $-24x = 0$ $\quad x = 0$

$f'(x)$ DNE when $x = \pm 4$

Intervals	$(-\infty,-4)$	$(-4,0)$	$(0,4)$	$(4,\infty)$
$f'(x)$	+	+	−	−
$f(x)$	↑	↑	↓	↓

Local Max at $x=0$

$f(0) = \frac{-4}{-16} = \frac{1}{4}$.

7d) Determine the intervals where $f(x)$ is concave up and concave down. Find any inflection points.

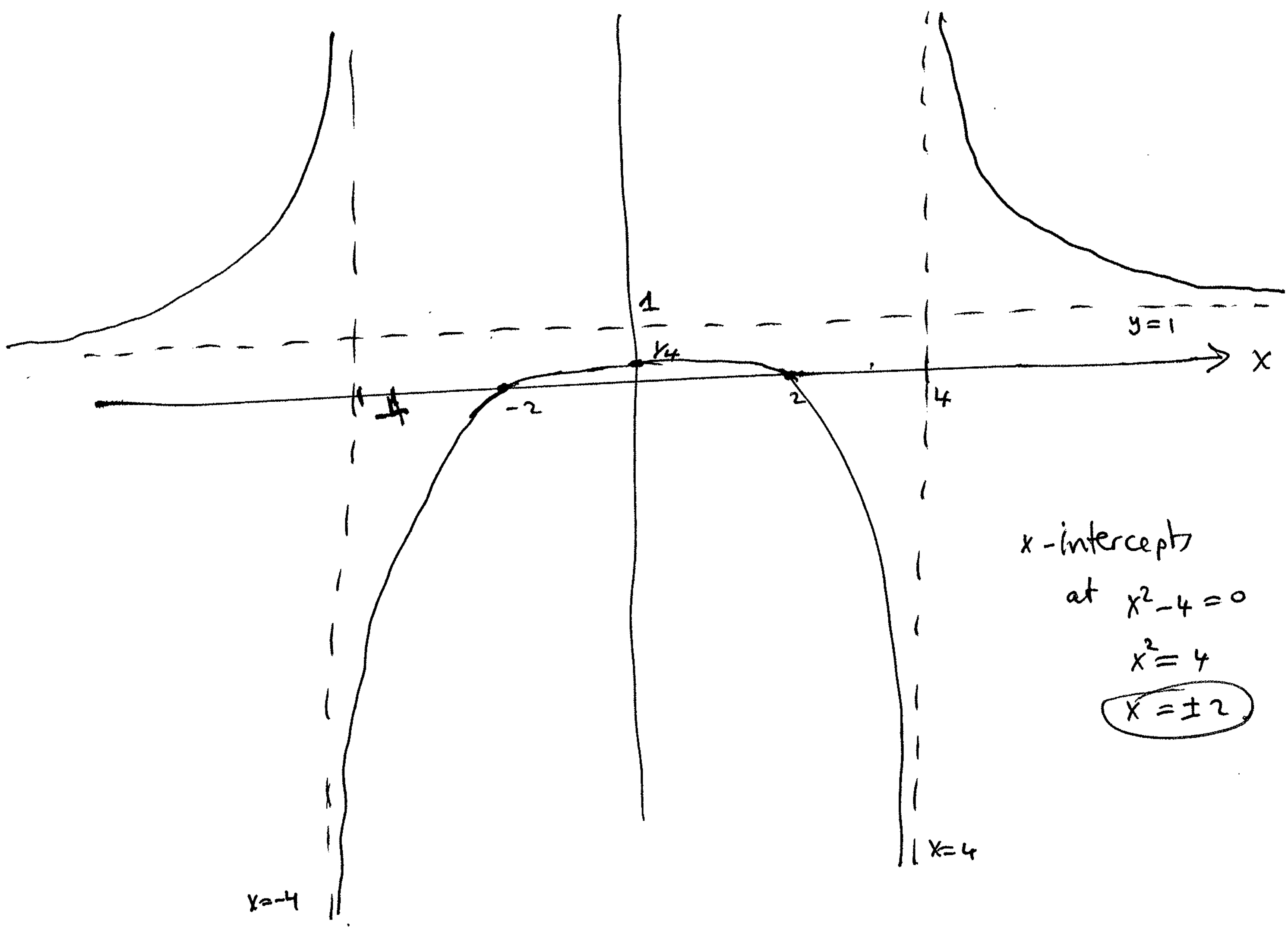
$f''(x) = 0$ $\quad 3x^2+16=0$ $\quad$ No solution!!

$f''(x)$ DNE $\quad$ at $x=-4, 4$.

intervals	$(-\infty,-4)$	$(-4, 4)$	$(4,\infty)$
$f''(x)$	$+$	$-$	$+$
$f(x)$	CU	CD	CU

No inflection points — changes in concavity at ± 4
→ not in domain of f!

7e) Sketch the graph of the function $f(x)$ from the previous page, using the information from parts (a) - (d). Clearly label horizontal and vertical asymptotes, local maximum and minimum points, and inflection points if they exist. You may also want to plot a few points.



x-intercepts at $x^2-4=0$

$x^2 = 4$

$x = \pm 2$