Fa'13: MATH 1914–030 **Differential and Integral Calculus I** **Noel Brady**

Monday 09/23/2013 Midterm I 1:30pm–2:45pm

Name:

Student ID:

Instructions.

1. Attempt all questions.
2. Do not write on back of exam sheets. Extra paper is available if you need it.
3. Show all the steps of your work clearly.

SOLUTIONS

Question	Points	Your Score
Q1	10	
Q2	15	
Q3	15	
Q4	15	
Q5	15	
Q6	15	
Q7	15	
TOTAL	100	

Q1]... [10 points] Suppose f is a function which is differentiable at the input a. Write down the limit definition of the derivative $f'(a)$.

$$f'(a) = \lim_{h \to 0} \left(\frac{f(a+h) - f(a)}{h} \right)$$

OR $$f'(a) = \lim_{x \to a} \left(\frac{f(x) - f(a)}{x - a} \right)$$

Write down two interpretations of the derivative $f'(a)$.

(1) Slope of tangent line to graph $y = f(x)$ at the point $(a, f(a))$.

(2) Instantaneous rate of change of $f(x)$ with respect to x at the input a.

Q2]...[15 points] For each of the two functions below, say if the function is continuous at 0, and also say if the function is differentiable at 0.

$$f(x) = \begin{cases} x\sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Continuous at 0

$\lim_{x \to 0} f(x) = 0 = f(0)$ by the squeeze theorem and the fact that

$-|x| \leq f(x) \leq |x|$ for all $x \neq 0$.

Not diff at 0

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{f(h)}{h} = \lim_{h \to 0} \frac{\cancel{h} \sin \frac{1}{h}}{\cancel{h}}$$

$$= \lim_{h \to 0} \left(\sin\left(\frac{1}{h}\right)\right)$$

Does Not Exist

(class notes)

$$g(x) = \begin{cases} 2x+1 & \text{if } x \geq 0 \\ x^2 & \text{if } x < 0 \end{cases}$$

$$\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} (2x+1) = 2(0) + 1 = 1$$

$$\lim_{x \to 0^-} g(x) = \lim_{x \to 0^-} (x^2) = 0^2 = 0$$

$\lim_{x \to 0^+} g(x) \neq \lim_{x \to 0^-} g(x) \Rightarrow \lim_{x \to 0} g(x)$ does not exist

$\Rightarrow g(x)$ not continuous at 0

$\Rightarrow g(x)$ not diff at 0.

(by Th^m in class: Diff. $\Rightarrow$ Cty)

Q3]...[15 points] Compute the following limit. Show all the steps of your work.

$$\lim_{x\to 3} \frac{\frac{1}{x^2} - \frac{1}{9}}{x-3}$$

$$= \lim_{x\to 3} \frac{\frac{9 - x^2}{9x^2}}{x-3}$$

$$= \lim_{x\to 3} \frac{(3-x)(3+x)}{9x^2\ (x-3)}$$

$$= \lim_{x\to 3} \frac{\overset{-1}{\cancel{(3-x)}}(3+x)}{9x^2\ \underset{1}{\cancel{(x-3)}}} \quad \cdots \text{ O.K. since } x-3 \neq 0$$

$$= \lim_{x\to 3} \frac{-(3+x)}{9x^2}$$

$$= \frac{-(3+3)}{9.3^2} = -\frac{6}{81} = \frac{-2}{27}$$

The limit above is the derivative $f'(a)$ of some function $f(x)$ at some input value a. Write down the function $f(x)$ and the input a.

$$f(x) = \frac{1}{x^2} \quad ; \quad a = 3.$$

Q4]...[15 points] Compute the following limit. Show all the steps of your work.

$$\lim_{h \to 0} \frac{\sqrt{4+h}-2}{h}$$

$$= \lim_{h \to 0} \left(\frac{\sqrt{(4+h)} - 2}{h} \right) \left(\frac{\sqrt{(4+h)} + 2}{\sqrt{4+h} + 2} \right)$$

$$= \lim_{h \to 0} \frac{(4+h) - 2^2}{h\left(\sqrt{4+h} + 2\right)}$$

$$= \lim_{h \to 0} \frac{\cancel{4} + h - \cancel{4}}{h\left(\sqrt{4+h} + 2\right)}$$

$$= \lim_{h \to 0} \frac{\cancel{h}^{\,1}}{{}_1\cancel{h}\left(\sqrt{4+h} + 2\right)} \quad \text{--- O.K. since } h \neq 0$$

$$= \frac{1}{\sqrt{4+0} + 2} = \frac{1}{2+2} = \frac{1}{4}$$

The limit above is the derivative $g'(b)$ of some function $g(x)$ at some input value b. Write down the function $g(x)$ and the input b.

$$g(x) = \sqrt{x} \quad ; \quad b = 4$$

Q5]... **[15 points]** Using the limit definition of the derivative, compute the derivative $f'(x)$ of the function $f(x) = x^2 + 3$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

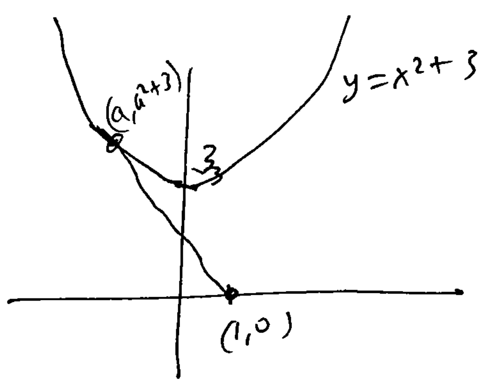
$$= \lim_{h \to 0} \frac{((x+h)^2 + 3) - (x^2 + 3)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2hx + h^2 + \cancel{3} - \cancel{x^2} - \cancel{3}}{h}$$

$$= \lim_{h \to 0} \frac{(2x+h)\cancel{h}^{1}}{\cancel{h}_{1}} \quad \ldots \quad \begin{matrix} \text{ok} \\ h \neq 0 \end{matrix}$$

Find points $(a, f(a))$ on the graph of the function $f(x) = x^2 + 3$ above at which the tangent line to the graph also passes through the point $(1, 0)$.

$$= \lim_{h \to 0} (2x+h) = 2x + 0 = \boxed{2x}.$$



Slope of line from (a, a^2+3) to $(1,0)$ = tangent line slope at (a, a^2+3)

$$\frac{(a^2+3) - 0}{(a) - 1} = 2(a)$$

$$\Rightarrow \frac{a^2+3}{a-1} = 2a \quad \Rightarrow \quad a^2 + 3 = 2a^2 - 2a$$

$$\Rightarrow a^2 - 2a - 3 = 0$$

$$\Rightarrow (a-3)(a+1) = 0$$

$a = 3, \Rightarrow a^2+3 = 12$

$a = -1 \Rightarrow a^2+3 = 4$

$(3, 12)$ & $(-1, 4)$

Q6]... **[15 points]** State the intermediate value theorem.

Suppose the function $f(x)$ is continuous at all points of the interval $[a,b]$ and suppose $f(a) \neq f(b)$.
Then for each N strictly between $f(a)$ and $f(b)$ there exists a number c in (a,b) such that $f(c) = N$.

Using the intermediate value theorem, or otherwise, show that there is an input between 0 and π at which the functions $y = \sin(x)$ and $y = x - 1$ have the same output values.

Let $f(x) = \sin(x) - (x-1)$

Note $\sin(x)$ and $(x-1)$ are continuous
(class notes: trig & poly functions)

$\Rightarrow$ $f(x)$ is continuous (difference of cts functions)

$f(0) = \sin(0) - (0-1) = 1 > 0$

$f(\pi) = \sin(\pi) - (\pi - 1) = 1 - \pi < 0$

0 lies between $f(0)$ & $f(\pi)$

Int. Val. Thm $\Rightarrow$ $f(c) = 0$ for some c in $(0,\pi)$.

ie $\sin(c) - (c-1) = 0$ for some c in $(0,\pi)$.

ie. $\sin(c) = c - 1$ for some c in $(0,\pi)$.

Q7]. . . [**15 points**] Write down the value of the following limits:

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1 \qquad \text{and} \qquad \lim_{x\to 0} \frac{\cos(x)-1}{x} = 0$$

Write down the angle addition formula for the sine function:

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

Use the limit definition of the derivative to compute the derivative $f'(x)$ of the function $f(x) = \sin(x)$. Show all the steps in your work.

$$f'(x) = \lim_{h\to 0}\left(\frac{f(x+h) - f(x)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{\sin(x+h) - \sin(x)}{h}\right)$$

used Angle Add^n formula

$$= \lim_{h\to 0}\left(\frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h}\right)$$

$$= \lim_{h\to 0}\left(\sin(x)\left(\frac{\cos(h)-1}{h}\right) + \cos(x)\left(\frac{\sin(h)}{h}\right)\right)$$

Limit Laws

$$= \sin(x)\lim_{h\to 0}\left(\frac{\cos(h)-1}{h}\right) + \cos(x)\lim_{h\to 0}\left(\frac{\sin(h)}{h}\right)$$

Limits at top of page

$$= \sin(x).\ 0 + \cos(x).\ 1$$

$$= \cos(x) \Rightarrow \boxed{\frac{d}{dx}(\sin(x)) = \cos(x)}$$

Bonus Question]... Suppose that $f(x)$ is differentiable at a.
Show that

$$\lim_{h\to 0} \frac{f(a)-f(a-h)}{h}$$

exists and is equal to $f'(a)$.

Let $k = -h$ Note as $h \to 0$ then $k \to 0$ too.

$$\lim_{h\to 0} \frac{f(a)-f(a-h)}{h} = \lim_{k\to 0}\left(\frac{f(a)-f(a+k)}{-k}\right) = \lim_{k\to 0}\left(\frac{f(a+k)-f(a)}{k}\right)$$

$$= f'(a) \quad \text{by def}^{\underline{n}}!$$

Show that

$$\lim_{h\to 0} \frac{f(a+h)-f(a-h)}{2h}$$

exists and is equal to $f'(a)$.

$$\lim_{h\to 0}\left(\frac{f(a+h)-f(a-h)}{2h}\right) = \frac{1}{2}\lim_{h\to 0}\left(\frac{f(a+h)-f(a-h)}{h}\right)$$

$$= \frac{1}{2}\lim_{h\to 0}\left(\frac{f(a+h)-f(a)+f(a)-f(a-h)}{h}\right) = \frac{1}{2}\lim_{h\to 0}\left(\frac{f(a+h)-f(a)}{h} + \frac{f(a)-f(a-h)}{h}\right)$$

$$= \frac{1}{2}\left(f'(a) + f'(a)\right)$$

$$= f'(a)$$

(1st limit by def$^{\underline{n}}$)

(2nd by comp. above)

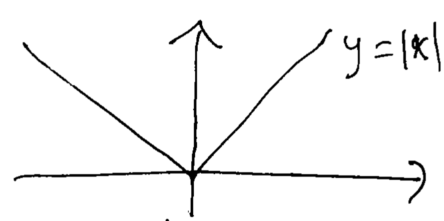
Suppose that g(x) is a function and that the limit

$$\lim_{h\to 0} \frac{g(a+h)-g(a-h)}{2h}$$

exists, and is some finite number L. Does his imply that $g(x)$ is differentiable at a, and that $g'(a) = L$?

<u>No</u> – $(a+h)$ & $(a-h)$ are symmetric inputs about a.
Non diff. function could exploit this symmetry.

~~eg~~ $g(x) = |x|$; $a = 0$.



$g'(0)$ does not exist

yet $\lim_{h\to 0}\left(\frac{g(0+h)-g(0-h)}{2h}\right)$

$$= \lim_{h\to 0}\left(\frac{|h|-|h|}{2h}\right) = \lim_{h\to 0}\left(\frac{0}{2h}\right) = 0$$