(People had trouble with the "logic" of this argument)

EXTRA Hwk IV $\quad$ Q2

2(a) $\quad g(x) = f(a) + f'(a)(x-a) + M(x-a)^2$

$g(a) = f(a)$ (i) $\quad g(a) = f(a) + f'(a)\cancel{(a-a)}^{\to 0} + M\cancel{(a-a)^2}^{\to 0}$

$= f(a)$ ✓

$g'(x) = 0 + f'(a).1 + 2M(x-a)$

$g'(a) = f'(a)$ (ii) $\quad g'(a) = f'(a).1 + 2M\cancel{(a-a)}^{\to 0}$

$= f'(a)$ ✓

2(b) $\quad g''(x) = 0 + 2M.1$

$= 2M$

$\boxed{g''(x) = 2M}$ — (iii)

2(c) choose M so that $g(b) = f(b)$.

2(d) $h(x) = f(x) - g(x)$ is twice differentiable on $[a,b]$

because it is the difference of $f(x)$ (which we are told is twice differentiable) and the polynomial $g(x)$ (which is many times differentiable).

2(e) $h(a) = f(a) - g(a) = f(a) - f(a)$ --- by (i) above in 2(a).

$= 0$

$h(b) = f(b) - g(b) = f(b) - f(b)$ .... by choice of M in 2(c).

$= 0$

MVT $\Rightarrow$ (Rolle's) There exists a point c_1 in (a, b) for which

$$h'(c_1) = 0.$$

2(f) Now look at $h'(x)$ on the interval $[a, c_1]$.

$h'(a) = f'(a) - g'(a) = f'(a) - f'(a)$ --- by (ii) above in 2(a)

$= 0$

$h'(c_1) = 0$ by 2(e) above

MVT $\Rightarrow$ (Rolle's) There is a point c_2 in (a, c_1) so that

$$(h')'(c_2) = 0$$

i.e. $h''(c_2) = 0$

From 2(f) we have $h''(c_2) = 0$.

2(g)

i.e. $f''(c_2) - g''(c_2) = 0$

i.e. $f''(c_2) - 2M = 0$ ... by (iii) above in 2(b).

$\Rightarrow M = \dfrac{f''(c_2)}{2}$ for some c_2 in (a, c_1) & hence in (a, b).

i.e.

Thus $g(x) = f(a) + f'(a)(x-a) + M(x-a)^2$

$= f(a) + f'(a)(x-a) + \dfrac{f''(c_2)}{2}(x-a)^2$

has the property that $g(b) = f(b)$ ...... from 2(c)

i.e. $\boxed{f(b) = f(a) + f'(a)(b-a) + \dfrac{f''(c_2)}{2}(b-a)^2}$

for some c_2 in (a, b).

This is what we wanted to establish (II).