Math 1914. Extra Hwk III Baby Steps to the Chain Rule!

The Derivative as a Linear Approximation.

- We saw in class that if $F'(a)$ exists, then the following expression holds

$$F(x) \;=\; F(a) + F'(a)(x-a) + \epsilon.(x-a) \qquad (I)$$

where $\epsilon \to 0$ as $x \to a$.

- We also saw that if the function $F(x)$ satisfies the following

$$F(x) \;=\; F(a) + L(x-a) + \epsilon.(x-a) \qquad (II)$$

for some number L and where $\epsilon \to 0$ as $x \to a$, then the function is differentiable at the input a and $F'(a) = L$.

- So the expression (II) together with the condition that $\epsilon \to 0$ as $x \to a$ is another way to say that the function $F(x)$ is differentiable at the input a. Notice that there are no "difference quotients" in sight.

- So if $F(x)$ is differentiable at the input a the tangent line $y = F(a)+F'(a)(x-a)$ approximates $f(x)$ with an error term of $\epsilon.(x-a)$ which tends to 0 as $x \to a$ faster than $(x-a)$ tends to 0. We see this because $\epsilon.(x-a)$ is the product of two quantities that are both going to 0 as $x \to a$, and one of them is $(x-a)$.

We recall the hypotheses and conclusion of **The Chain Rule.**
Hypotheses (what we are given):

(i) The function $f(u)$ is differentiable at the input $g(a)$ with derivative $f'(g(a))$; and

(ii) The function $g(x)$ is differentiable at the input a with derivative $g'(a)$.

Conclusion: The composite function $(f \circ g)(x) = f(g(x))$ is differentiable at the input a with derivative

$$(f \circ g)'(a) \;=\; f'(g(a)).g'(a)$$

Our goal in these exercises is to explore why the chain rule is true, and to understand intuitively why the derivative of a composite of two functions is equal to the product of the derivatives of the component functions. We do so in a series of steps.

Step 1. Composition of two straight line functions. Start with two straight line functions $y = \ell_1(u) = 2u+3$ and $u = \ell_2(x) = 5x-4$. Write down an expression for the composite function $y = (\ell_1 \circ \ell_2)(x)$. [Hint: Since we have used the intermediate variable u, it will be an easy matter of substituting one expression for u in terms of x into the other expression for y in terms of u.]

There are two observations you can make about this composite function.

1. The composite is also a straight line function; and

2. The slope of the composite line is related to the slopes of the original two lines in a straightforward manner. How are these related?

Step 2. Functions and their Tangent Lines.

(a) Suppose $u = g(x)$ is differentiable at the input a. Show that the equation of the tangent line to the graph of $g(x)$ at the point $(a, g(a))$ is given by

$$u \;=\; g'(a)(x-a) + g(a) \qquad (A)$$

Draw a typical graph of $u = g(x)$ and its tangent line $u = g'(a)(x-a) + g(a)$ in the xu-plane.

(b) Suppose $y = f(u)$ is differentiable at the input $g(a)$. Show that the equation of the tangent line to the graph of $f(u)$ at the point $(g(a), f(g(a)))$ is given by

$$y \;=\; f'(g(a))(u - g(a)) + f(g(a)) \qquad (B)$$

Draw a typical graph of $y = f(u)$ and its tangent line $y = f'(g(a))(u-g(a)) + f(g(a))$ in the uy-plane.

Step 3. Composition of Tangent Lines. Substitute the expression for u in equation (A) into equation (B) to write out the composite of the two tangent lines in Step 2 above. Verify that you get the following expression.

$$y \;=\; f(g(a)) + f'(g(a)).g'(a).(x-a)$$

So we see the expression $f'(g(a)).g'(a)$ appears as the slope of the composition of the tangent lines, just as in Step 1. Now we have to compose the two functions $y = f(u)$ and $u = g(x)$ and verify that the composition is differentiable at the input a and that the derivative (and hence the tangent line) is as the Chain Rule states.

Step 4. Composition of two Functions Expressed as Approximations to their Tangent Lines (Proof of the Chain Rule).

(a) Use the formulation of differentiability given in (I) and the fact that $g(x)$ is differentiable at the input a, to get the following

$$u \;=\; g(x) \;=\; g(a) + g'(a).(x-a) + \epsilon_1.(x-a) \qquad (C)$$

where $\epsilon_1 \to 0$ as $x \to a$.

(b) Use the formulation of differentiability given in (I) and the fact that $f(u)$ is differentiable at the input $g(a)$, to get the following

$$y \;=\; f(u) \;=\; f(g(a)) + f'(g(a)).(u-g(a)) + \epsilon_2.(u-g(a)) \qquad (D)$$

where $\epsilon_2 \to 0$ as $u \to g(a)$.

(c) Let u be given by equation (C). Verify that as $x \to a$, then $u \to g(a)$.

(d) Substitute the expression for u in (C) into equation (D) and verify that you get the following

$$y \;=\; (f \circ g)(x) \;=\; f(g(x)) \;=\; f(g(a)) + f'(g(a)).g'(a).(x-a) + \epsilon_3.(x-a) \qquad (E)$$

The term ϵ_3 will actually be a sum of three product terms. Write it out explicitly and verify that $\epsilon_3 \to 0$ as $x \to a$.

(e) Compare (E) and the property of ϵ_3 with (II). What can you conclude about the composite function $(f \circ g)(x)$?