MATH 2513-001 Discrete Mathematics Fall 2009

Symmetries of the plane, rigid motions

1. The *euclidean plane* $\mathbb{E}^2$ is just the usual cartesian plane $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ with some euclidean geometry added in to the mix. This is commonly done by using Pythagoras' Theorem to tell one how to compute distances between points

$$d((x, y), (a, b)) \;=\; \sqrt{(x-a)^2 + (y-b)^2}$$

This is not anything new or scary. It is just the usual "distance formula" from coordinate geometry. It is good to realize that this is how euclidean geometry enters the picture.

2. An *isometry* (or *rigid motion*) of the plane is just a bijective map of the plane which preserves distances. That is, it is a bijective map $f : \mathbb{E}^2 \to E^2$ with the property that

$$d(f(x, y), f(a, b)) \;=\; d((x, y), (a, b))$$

for all points (x, y) and (a, b) in $\mathbb{E}^2$.

3. **Exercise.** Show that any map $f : \mathbb{E}^2 \to \mathbb{E}^2$ which preserves distances is a bijection. So, we do not need to use the word bijective in definition 2 above.

4. **Exercise.** Show that the set of isometries of the plane form a group under composition. This group is denoted by $\mathrm{Isom}(\mathbb{E}^2)$ and is called the *euclidean isometry group*. It is a subgroup of the group $\mathrm{Perm}(\mathbb{E}^2)$.

5. **Exercise.** Show that an isometry is completely determined by where it sends three non collinear points.

6. **Example.** A *reflection* in a line l is defined as follows. Given a point (x, y) firs draw the unique perpendicular m to l. Next define $l(x, y)$ to be (x, y) if (x, y) lies on the line l, otherwise define it to be the unique point on m which is different from (x, y) and whose distance from l equals the distance from (x, y) to l.

   Show that a reflection in a line is an isometry.

7. **Example.** A rotation through angle θ about the point P takes a point Q to the point Q' so that the angle QPQ' measures θ counterclockwise.

   Show that a rotation is an isometry of $\mathbb{E}^2$.

8. **Example.** Let $\mathbf{v} = \langle a, b\rangle$ be a vector in the plane. A *translation* through $\mathbf{v}$ is defined to be the map

$$(x, y) \;\mapsto\; (x + a, y + b)$$

9. **Example.** Let l be a line in $\mathbb{E}^2$ and $\mathbf{v}$ be a vector in $\mathbb{E}^2$ which is parallel to l. A *glide reflection* with axis the line l and translation component $\mathbf{v}$ is defined to be the composition of translation through $\mathbf{v}$ followed by reflection in l.

10. **Exercise.** Show that the composition of reflections in two lines which intersect in a point P equals a rotation about P through twice the angle between the lines. Hint: it suffices to show that this is the behavior of the composition on three non collinear points.

11. **Exercise.** Show that the composition of reflections in two parallel lines is a translation through a vector which is perpendicular to the lines, and whose length equals twice the distance between the lines. The vector points in the direction from the first to the second line (in order of composition). Hint: it suffices to show that this is the behavior of the composition on three non collinear points.

12. **Exercise.** Show that every isometry is given by composition of reflections in at most three lines. Hint: see what your given isometry does to three non collinear points. Now show that a triangle can be mapped onto any other triangle which is congruent to it by composition of reflections in at most three lines.

13. **Exercise.** Show that there are 4 types of isometries of $\mathbb{E}^2$: rotations, translations, reflections and glide reflections. The identity element $\mathbb{I}_{\mathbb{E}^2}$ can be considered as either a rotation (through angle of 0) or a translation (through a distance of 0).

14. **Coordinates.** Show that reflection in the line through the origin and making angle θ with the positive x-axis is given by

$$(x, y) \mapsto (x\cos(2\theta) + y\sin(2\theta), x\sin(2\theta) - y\cos(2\theta))$$

15. **Coordinates.** Show that rotation through θ about the origin is given by

$$(x, y) \mapsto (x\cos(\theta) - y\sin(\theta), x\sin(\theta) + y\cos(\theta))$$

16. **Coordinates.** Show that every element of $\mathrm{Isom}(\mathbb{E}^2)$ can be written out explicitly in coordinates either as

$$(x, y) \mapsto (x\cos(2\theta) + y\sin(2\theta) + a, x\sin(2\theta) - y\cos(2\theta) + b)$$

for suitable angle θ and numbers a and b, or as

$$(x, y) \mapsto (x\cos(\theta) - y\sin(\theta) + a, x\sin(\theta) + y\cos(\theta) + b)$$

for suitable angle θ and numbers a and b.

17. **Coordinates.** Use coordinates to check that the composition of two reflections in lines through the origin is indeed a rotation about the origin. Use the formula for a reflection given in part 14 for two angles α and β. You'll have to also use the formula for a rotation given in part 15, and you will have to remember your addition/subtraction laws for sine and cosine from trig.

18. **Coordinates.** Use coordinates to check that the composition of two rotations about the origin is again a rotation about the origin. Use the formula for rotations given in 15 for two angles α and β. You'll have to remember your addition/subtraction laws for sine and cosine from trig.

19. **Symmetry groups of (regular) polygons.** Let $P_n \subset \mathbb{E}^2$ be a regular polygon with n sides. So P_3 is an equilateral triangle, P_4 a square, P_5 a regular pentagon, etc

    The group of all rigid motions of $\mathbb{E}^2$ which take P_n to itself is denoted by $\mathrm{Symm}(P_n)$ and also by D_n and is called the *dihedral group* of order $2n$. Note that D_n is a subgroup of $\mathrm{Isom}(\mathbb{E}^2)$.

    Show that the elements of D_n must all fix the center O of P_n. Show that all the elements of D_n are either rotations about O or reflections in lines through O. Show that D_n has precisely $2n$ elements: n rotations and n reflections. By labeling the vertices of D_n with the integers $1, \ldots, n$ find an explicit map from D_n to a suitable subgroup of $\mathrm{Perm}(\{1, \ldots, n\})$ in the cases $n = 3, 4, 5$.

    In the case $n = 4$, use a labeling of the edges of P_4 to find another copy of D_4 inside of $\mathrm{Perm}(\{1, 2, 3, 4\})$.

20. **Exercise.** Think about how the above arguments would work for the space $\mathbb{E}^3$ which is $\mathbb{R}^3$ with the distance function

$$d((x, y, z), (a, b, c)) = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}$$

    Think about the symmetries of a regular tetrahedron (pyramid with triangular base) or of a regular cube.