Sample Questions from Number Theory

State the Division Algorithm.

State the well ordering (W.O.) principle.

Define \( \gcd(a,b) \).

State the Fundamental Theorem of Arithmetic.

Let \( a, b \in \mathbb{Z}, \ b > 0 \). Use W.O. to prove the existence of integers \( q, r \) so that \( a = bq + r, \ 0 \leq r < b \).

Let \( a, b \in \mathbb{Z}^+ \). Use W.O. to prove that \( (\text{the least } \text{ term}) \cdot \gcd(a,b) \) can be written as \( \gcd(a,b) = sa + tb \) for some \( s, t \in \mathbb{Z} \).

Define \( a \mid b \).

Prove that if \( \gcd(a,b) = 1 \) and \( a \mid bc \), then \( a \mid c \).

Use Euclidean Algorithm to find \( \gcd(3b,6b) \) and to find integers \( s, t \) so that \( \gcd(3b,6b) = s(3b) + t(6b) \).

Use Fund Thm to prove that \( \sqrt{6} \notin \mathbb{Q} \).

Use Fund Thm to prove that the only integers whose square roots are rational are the perfect squares, (same for cubes & cube roots).

Use Fund Thm to prove that \( \log_2(10) \notin \mathbb{Q} \).

If \( \frac{a^2}{b^2} = 2 \), then show that \( \frac{2b - a}{a - b} = \frac{a}{b} \). Use this fact and W.O. to prove that \( \sqrt{2} \notin \mathbb{Q} \).
Prove that there are primes \( > n \) for every positive integer \( n \) as follows: consider \( n! + 1 \).

Prove that \( \gcd(a, a+1) = 1 \) for any \( a \in \mathbb{Z}^+ \).

Prove that \( \gcd(a, a+2) = 1 \) or \( 2 \) for \( a \in \mathbb{Z}^+ \) (when do you get 1, when 2?).

Suppose that \( \gcd(a, b) = 1 \). Prove that \( \gcd(a+b, a-b) = 1 \) or 2.

Suppose \( k \in \mathbb{Z}^+ \). Show that \( 6k-1, 6k+1, 6k+2, 6k+3, 6k+5 \) are all pairwise relatively prime.

Let \( a, b, c \in \mathbb{Z}^+ \). Show that \( \gcd(a, b, c) \) can be expressed as a combination \( sa + tb + uc \) for some \( s, t, u \in \mathbb{Z} \).

Use Euclidean Algorithm to compute
\[
\begin{align*}
gcd(102, 222) \\
gcd(20785, 4350) \\
gcd(51, 87) \\
gcd(981, 1234)
\end{align*}
\]
Write out prime factorization of \( 10! \), \( 20! \).
How many zeros at end \( 1000 \)?